Graph Theory Applications EPFL, Spring 2014

Solution to Problem Set 3

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Problem 1. Let $C_1, \cdots, C_k$ denote the connected components of G. We can assume that $k > 1$, otherwise G is connected and there is nothing to prove. Thus, by definition of connectedness, it suffices to find a path between any two vertices in G^c. Note that for any two vertices belonging to different connected components of G, there is an edge in G^c. Moreover, if u and v belong to the same connected component C_i, then there is path of length 2 between u and v via a vertex w belonging to another connected component C_j, with $j \neq i$.

Problem 2. Let $\pi = \{\pi(1), \pi(2), \ldots, \pi(n)\}$. It is easy to show that left multiplying a matrix A by P results in permuting the rows of A according to π, i.e., row i of A becomes row $\pi(i)$ of PA, and right multiplying A by P^T results in permuting the columns of A according to π, i.e., column j of A becomes column $\pi(j)$ of AP^T . Therefore, the (i, j)-th entry of A, henceforth denoted as $A_{i,j}$, appears as the $(\pi(i), \pi(j))$-th entry of PAP^T .

Now suppose that $A_H = PA_G P^T$. Then we know that for all i, j,

$$A_{H_{i,j}} = (PA_G P^T)_{i,j} \implies A_{H_{\pi(i),\pi(j)}} = (PA_G P^T)_{\pi(i),\pi(j)} = A_{G_{i,j}},$$

where the last equality follows from the discussion above. We conclude that vertices i and j of G are adjacent if and only if vertices $\pi(i)$ and $\pi(j)$ of H are adjacent (since the corresponding entries of the adjacency matrices are the same). Then, letting $\theta(i) = \pi_i$ map the vertices of G to the vertices of H we conclude that two vertices i and j are adjacent in G if and only if their images $\theta(i)$ and $\theta(j)$ are adjacent in H, and therefore G and H are isomorphic. In other words, G and H are the same graph with the vertices renumbered.

Problem 3. We need to show that

$$\det(A - kI) = 0$$

We have that

$$\begin{aligned}
\det(A - kI) &= \det \begin{pmatrix} a_{1,1} - k & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} - k & \ldots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} - k \end{pmatrix} \\
&= \det \begin{pmatrix} (\sum_{j=1}^n a_{1,j}) - k & a_{1,2} & \ldots & a_{1,n} \\ (\sum_{j=1}^n a_{2,j}) - k & a_{2,2} - k & \ldots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ (\sum_{j=1}^n a_{n,j}) - k & a_{n,2} & \ldots & a_{n,n} - k \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & a_{1,2} & \ldots & a_{1,n} \\ 0 & a_{2,2} - k & \ldots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n,2} & \ldots & a_{n,n} - k \end{pmatrix} = 0,
\end{aligned}$$

where the second equality follows from properties of the determinant and the third from the fact that in a k-regular graph, every vertex i has exactly k neighbors, therefore each row (and each column) of A sum up to k.

Problem 4. For the "if" part, we want to show that $\det(A + kI) = 0$. If G is bipartite, then we can relabel the vertices such that A looks like

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where B is a square matrix (B is square from regularity). So, $A + kI$ has the following structure:

$$A + kI = \left(\begin{array}{ccccc|ccccc} k & 0 & & \dots & 0 & & & & & \\ 0 & k & 0 & \dots & 0 & & & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots & & & B & & \\ 0 & \dots & 0 & k & 0 & & & & & \\ 0 & & \dots & 0 & k & & & & & \\ & & & & & k & 0 & & \dots & 0 \\ & & & & & 0 & k & 0 & \dots & 0 \\ & & B^T & & & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & & 0 & \dots & 0 & k & 0 \\ & & & & & 0 & & \dots & 0 & k \end{array} \right).$$

Take the vector

$$v = (\underbrace{1, \dots, 1}_{n/2}, \underbrace{-1, \dots, -1}_{n/2}).$$

From the regular property it is clear that

$$v(A + kI) = 0,$$

so v is a non-trivial linear combination of the rows which results 0, thus $\det(A + kI) = 0$.

For the "only if" part, we know that there exists a v for which

$$v(A + kI) = 0. \tag{1}$$

We can also assume that the largest (absolute value) element of v is 1. (If not, we divide the vector by its largest element.) Let 1 be the i-th element of v. Consider the i-th column of $A' = A + kI$. We know that the diagonal elements of A' are at least k, so we have from (1) that

$$\sum_{j=1, j \neq i}^{n} v_j {A'}_{j,i} \leq -k. \tag{2}$$

Also, from regularity,

$$\sum_{j=1, j \neq i}^{n} {A'}_{j,i} \leq k. \tag{3}$$

Since $v_i = 1$ is the largest element in v, we must have that (2) and (3) are equalities and $v_j = -1$, if $A'_{j,i} \neq 0$. In other words, vertex i doesn't have self-loops and every vertex j that is connected to i has $v_j = -1$.

Consider a j for which (i, j) is an edge. We apply the same argument on the j-th column of A'. The same holds, only the sign switch,

$$\sum_{\ell=1, \ell \neq j}^{n} v_\ell {A'}_{\ell,j} \geq k.$$

Again, from regularity,

$$\sum_{\ell=1,\ell\neq j}^{n} {A'}_{\ell,j} \leq k,$$

so $v_\ell = 1$ for every ℓ for which (j, ℓ) is an edge. We can go on with the same reasoning and since G is connected we eventually give a constraint on every element of v being equal to 1 or -1 and none of the vertices can have self-loops. Thus every edge has the property that it connects two vertices (i, j) with $v_i = -v_j$. Consequently, the sign of the elements in v gives a partitioning of the vertices, hence G is bipartite.

Problem 5. The desired partition of the vertex set of G_{C} into two parts witnessing its bipartiteness looks as follows.

Let $V_1 = \{u \in \{0,1\}^n \mid u \text{ contains odd number of ones}\}$ and $V_2 = V \setminus V_1$. We show that there is no edge within V_1 and within V_2. Indeed, if there is an edge between u and v and the number of ones in u is odd, then the number of ones in v is even, because u and v must differ in exactly one component. Similarly, if there is an edge between u and v and the number of ones in u is even, then the number of ones in v is odd. This suffices to prove the claim.

Problem 6. Let S be the subset of $\mathbb{R}^3$ that consists of all vectors (x_1, x_2, x_3) such that x_i is non-negative for $i \in \{1, 2, 3\}$ and $x_1 + x_2 + x_3 = 1$. If there exists $x \in S$ such that $Ax = 0$, then we are done. Otherwise, we know that for every $x \in S$, Ax has non-negative entries, not all of them zero. Let us write $|Ax|$ for the sum of the coefficients of Ax. Then, the map $\phi(x) = Ax/|Ax|$ is a continuous map from S to S.

Now, geometrically S is a simplex of dimension 2, and therefore it is homeomorphic to a ball of dimension 2. The Brouwer fixed point theorem implies that ϕ has a fixed point, so there must be some $\tilde{x}$ such that $A\tilde{x} = |A\tilde{x}|\tilde{x}$. Hence, $\tilde{x}$ is an eigenvector with eigenvalue $|A\tilde{x}|$. Since $\tilde{x} \in S$, it has non-negative coefficients, so the result is proved.