Solution to Problem Set 1

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Problem 1. Frame the problem as a graph with a vertex for each person and an edge between two people if they have shaken hands. Clearly, there are n vertices in the graph and the number of people vertex i has shaken hands with corresponds to the degree of the graph. We claim that, the number of different degrees in the graph can be at most $n-1$. If there is a degree 0 vertex, then there cannot be a degree $n-1$ vertex, which makes the number of degrees equal to $n-2$. On the other hand, if there is no degree 0 vertex, there are at most $n-1$ degrees $(1,\ldots,n-1)$. Since there are n people, by the pigeonhole principle, there must be at least 2 people with the same degree, which proves the claim.

Problem 2. Consider a fixed color c_k, with $1 \leq k \leq 8$. Since we have 20 balls of c_k and they are all placed in the 6 jars, by the pigeonhole principle there exists at least one jar that contains at least two balls of color c_k. Clearly, this is true for all colors c_k, i.e.,

$$\text{For every } 1 \leq k \leq 8, \text{ there is a jar } j_k \text{ that contains at least 2 balls of color } c_k.$$

But there are only 6 jars, so by the pigeonhole principle there is a jar that appears at least twice in the set $\{j_1, j_2, \ldots, j_8\}$, and therefore contains at least two balls of two different colors.

Problem 3. By the binomial theorem we have that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = \sum_{i=0}^{n}(1)^{n-i}(-1)^i\binom{n}{i} = (1-1)^n = 0.$$

Using again the binomial theorem we have that

$$\sum_{i=0}^{n}\binom{n}{i} = 2^n.$$

By summing up these two equations, we obtain

$$2^n = \sum_{i=0}^{n}\binom{n}{i} + \sum_{i=0}^{n}(-1)^i\binom{n}{i} = 2\sum_{\substack{i=0\\ i\ even}}^{n}\binom{n}{i},$$

which suffices to prove the claim.

Problem 4.

1. Consider the graph G and let vertex i be the vertex with degree d_i. Let $S_1 \subset V(G)$ be the set of vertices which are connected to vertex 1, say $S_1 = \{n_1, \cdots, n_{d_1}\}$. Let $S_1^* \subset V(G)$ be the set of vertices with degrees $d_2, d_3, \cdots, d_{d_1+1}$, i.e., $S_1^* = \{2, \cdots, d_1+1\}$. From the graph G (which exists because $\mathbf{d}$ is graphical) we want to construct a graph G^* in which the neighbors of vertex 1 are the vertices in S_1^*.

   Suppose that S_1 and S_1^* are disjoint. If this is not the case, we just remove the common vertices from both sets and the proof works the same. Remove from the graph G all the edges departing from the vertex 1 and add edges that go from vertex 1 to vertices $2, 3, \cdots, d_1+1$, respectively.

This new graph, call it G_{int}, is s.t. the degree of vertex i is still d_i if this vertex does not belong neither to S_1 nor to S_1^*. Each of the vertices in S_1 loses one edge (and, hence, one degree), while each of the vertices in S_1^* gains one edge (and, hence, one degree).

Start from vertex 2 and pick one of his neighbors (different from vertex 1), call it $\bar{v}$, which is not a neighbor of vertex n_1. Note that $\bar{v}$ exists because $d_2 \geq d_{n_1}$ (the sequence $\mathbf{d}$ is nonincreasing) and in the original graph G the vertex 2 is not connected to vertex 1, while vertex n_1 is connected to vertex 1. Now, remove the edge from 2 to $\bar{v}$ and add an edge from $\bar{v}$ to n_1. Repeat this procedure for the couples of vertices $(j+1, n_j)$ for $j \in \{2, \cdots, d_1\}$ to obtain the graph G^*.

2. If $\mathbf{d}'$ is graphic, then there exists a graph G' with degree sequence $\mathbf{d}'$. Add a new vertex to G', which is connected to the vertices with degrees $d_2, d_3, \cdots, d_{d_1+1}$. This new graph, call it G, has degree sequence $\mathbf{d}$. Therefore $\mathbf{d}$ is graphic.

   To prove the other implication we are going to need the result of the previous point. Start from the graph G with degree sequence $\mathbf{d}$. Build the graph G^* with the procedure of point 1. Remove from G^* the vertex with degree d_1 which is connected to the vertices with degrees $d_2, d_3, \cdots, d_{d_1+1}$. This new graph, call it G', has degree sequence $\mathbf{d}'$. Therefore, $\mathbf{d}'$ is graphic.

3. Let f be the operator that from $\mathbf{d}$ produces $\mathbf{d}'$, i.e., it removes the biggest element of $\mathbf{d}$, call it d_1, and subtracts 1 from the following d_1 elements. Then, consider the following recursion:

$$\mathbf{d}_0 = \mathbf{d},$$
$$\mathbf{d}_{i+1} = f(\mathbf{d}_i), \qquad i \geq 0.$$

   Since $\mathbf{d}$ is graphic if and only if $f(\mathbf{d})$ is graphic, $\mathbf{d}_i$ is graphic for all i. If $\mathbf{d}_0$ is graphic, this recursion will converge to a sequence with a single element which is 0. Consider the previous sequence and build a graph with has that degree sequence (it is easy, since this graph has only two nodes…). Now, we want to go backwards in the recursion. Given a graph with degree sequence $\mathbf{d}_{i+1}$, in order to build a graph with degree sequence $\mathbf{d}_i$, one simply adds a vertex with degree given by the first element of $\mathbf{d}_i$ and connects it to the vertices with the highest degrees. We iterate this procedure until we get a graph with degree sequence $\mathbf{d}$, which is the desired graph.

   With some further passages, one can prove that if $\mathbf{d}$ fulfils the conditions (1), then the recursion converges to a sequence with a single element which is 0. This sequence is graphic and, hence, all the other sequences of the recursion are graphic. In other words, this means that the conditions (1) are also sufficient for a sequence to be graphic.

Problem 5. First of all, observe that

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u,v) = \max_{u \in V(G)} \max_{v \in V(G)} d(u,v) = \max_{u \in V(G)} \text{ecc}(u).$$

Since, by definition, $\text{rad}(G) = \min_{u \in V(G)} \text{ecc}(u)$, then

$$\text{rad}(G) \leq \text{diam}(G).$$

Now, let $x^* \in V(G)$ be the (or one of the) most "central" vertices in G, i.e., such that $\text{ecc}(x^*) = \text{rad}(G)$. By definition, $\text{ecc}(x^*) = \max_{u \in V} d(x^*, u)$. Therefore, for *every* vertex $u \in V(G)$,

$$d(x^*, u) \leq \text{ecc}(x^*).$$

For every pair of vertices (u, v), the shortest path between u and v cannot be longer than the path that goes from u to v through x^* (triangle inequality). Therefore,

$$d(u,v) \leq d(u,x^*) + d(x^*,v)$$

Combining the above two, we have

$$d(u,v) \leq d(u,x^*) + d(x^*,v) \leq 2 \cdot \mathrm{ecc}(x^*).$$

Let (a,b) be the (or one of the) pair of vertices such that $\mathrm{diam}(G) = d(a,b)$. Since the inequality above holds for every pair of vertices it also holds for (a,b), which proves the second inequality of the problem.

For $G = K_3$, $\mathrm{rad}(G) = \mathrm{diam}(G)$, and if G is a star, then $2 \cdot \mathrm{rad}(G) = \mathrm{diam}(G)$.

Problem 6. Choose a vertex x in $V(G)$ and an edge $xy \in E(G)$ and consider the sets S_i and their neighborhoods $N(S_i)$ where

$$S_0 = \{x,y\},\ S_1 = N(S_0)$$
$$S_{i+1} = N(S_i) \setminus (S_{i-1} \cup S_i) \text{ for } 1 \leq i \leq \mathrm{diam}(G) - 2.$$

Clearly, by the definition of diameter, $V(G) = \cup_i S_i$ and since the maximum degree is $\Delta(G)$,

$$V(G) \leq 2\left(1 + (\Delta(G)-1) + (\Delta(G)-1)^2 + \ldots + (\Delta(G)-1)^{\mathrm{diam}(G)-1}\right) = 2\frac{(\Delta(G)-1)^{\mathrm{diam(G)}} - 1}{\Delta(G)-2}.$$