# Problem Set 1

Date: 20.02.2014 Not graded

**Problem 1.** Suppose that $n$ people are attending a party and there are some handshakes between different people in the party. Show that there are at least two persons who have shaken hands with the same number of people.

*Hint. Pigeonhole.*

**Problem 2.** Balls of 8 different colors are placed in 6 jars. There are 20 balls of each color. Show that there must be a jar containing two pairs from two different colors of balls (for example, there is a jar containing at least two blue and at least two green balls).

**Problem 3.** Prove that for any $n \in \mathbb{N}$,

$$\sum_{\substack{i=0 \\ i\ even}}^{n} \binom{n}{i} = 2^{n-1},$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ denotes the binomial coefficient.

*Hint. There is a lot of ways in which one can prove the claim. One way is to show first that* $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$ *with the binomial theorem.*

**Problem 4.** Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be a nonincreasing sequence of non-negative integers. We say that $\mathbf{d}$ is *graphic* if there exists a simple graph with degree sequence $\mathbf{d}$. Recall that in class we discussed that for the sequence $\mathbf{d}$ to be *graphic* we need that

$$\begin{aligned} &\sum_{i=1}^{n} d_i \text{ is even}, \\ &\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \qquad 1 \le k \le n. \end{aligned} \tag{1}$$

The aim of this exercise is to show an algorithm to construct a graph with degree sequence $\mathbf{d}$, if such a graph exists.

1. Suppose that $\mathbf{d}$ is graphic, i.e., there exists a graph $G$ which has degree sequence $\mathbf{d}$. Show that there exists a graph $G^*$ s.t. the vertex with degree $d_1$ is connected to the vertices with degrees $d_2, d_3, \cdots, d_{d_1+1}$.

2. Let $\mathbf{d}' = (d_2 - 1, d_3 - 1, ..., d_{1+d_1} - 1, d_{2+d_1}, \cdots, d_n)$. Use the previous result to show that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

3. Now, what could be an algorithm which constructs a graph with degree sequence $\mathbf{d}$ (if such a graph exists)?

Note that with the procedure outlined in this exercise we can prove that the conditions (1) are also *sufficient* for the sequence $\mathbf{d}$ to be graphic.

For the next two problem we are going to need some definitions. Let $d(u, v)$ denotes the graph distance between the vertices $u$ and $v$ ($u, v \in V(G)$), which is the minimum length of the paths connecting them, i.e., the length of the shortest path.

Let diam$(G)$ denote the diameter of $G$, which is the longest shortest path between any two vertices of the graphs, i.e.,

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

Let $\text{ecc}(v)$ denote the eccentricity of the vertex $v \in V(G)$, which is defined as the maximum graph distance between $v$ and any other vertex $u \in V(G)$, i.e.,

$$\text{ecc}(v) = \max_{u \in V(G)} d(u, v).$$

Let $\text{rad}(G)$ denote the radius of the graph $G$, which is defined as the minimum graph eccentricity of any graph vertex in $G$, i.e.,

$$\text{rad}(G) = \min_{v \in V(G)} \text{ecc}(v).$$

Let $N(v)$ denote the neighborhood of a vertex $v \in V(G)$, which is defined as the set of all vertices adjacent to $v$ including $v$ itself. By extension, the neighborhood $N(S)$ of a set $S \subseteq V(G)$ of vertices is defined as the union of the neighborhoods of the vertices $v \in S$, i.e.,

$$N(S) = \bigcup_{v \in S} N(v).$$

**Problem 5.** Show that for every connected graph $G$ the following inequalities hold:

$$\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G).$$

Find a graph where rad$(G) =$ diam$(G)$, and a graph where 2rad$(G) =$ diam$(G)$.

**Problem 6.** If the maximum degree of a connected bipartite graph $G$ is $\Delta(G)$, prove that the maximum number of vertices in it is

$$|V(G)| \leq 2 \frac{(\Delta(G) - 1)^{\text{diam}(G)} - 1}{\Delta(G) - 2}.$$