Solution to Problem Set 12

Problem 1. Let Σ be the random permutation and let $X(\Sigma)$ be the number of fixed point of Σ. Then, we can express $X(\Sigma)$ as a sum of indicator variables,

$$X(\Sigma) = \sum_{i=1}^{n} X_i(\Sigma),$$

where $X_i(\Sigma) = 1$ if $\Sigma(i) = i$ and 0 otherwise. Then,

$$\mathbb{E}(X_i) = \mathbb{P}(\Sigma(i) = i) = \frac{1}{n}.$$

By linearity of expectation, we conclude that

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = 1.$$

Therefore, a random permutation has 1 fixed point on average.

Problem 2. Pick an inscribed regular pentagon uniformly at random (to do so, one can pick uniformly at random one point of the circle and then construct the only regular pentagon inscribed in the circle passing for that point). Let E_i denote the event that its i-th vertex ($i \in \{1, 2, \cdots, 5\}$) lies in the red part of the circle. Then, $\mathbb{P}(E_i) = 1/6$ and, consequently, we have that

$$\begin{aligned}\mathbb{P}(\text{random pentagon is black}) &= 1 - \mathbb{P}(\text{random pentagon has a red vertex}) \\ &= 1 - \mathbb{P}(\bigcup_{i=1}^{5} E_i) \geq 1 - \sum_{i=1}^{5} \mathbb{P}(E_i) = 1 - \frac{5}{6} > 0.\end{aligned}$$

Since $\mathbb{P}(\text{random pentagon is black}) > 0$, the existence of the black pentagon is guaranteed.

Problem 3. Let us calculate the expected number of Hamiltonian paths in a random tournament T (every edge has a random orientation, chosen independently with probability $1/2$). Pick a permutation σ on $\{1, \cdots, n\}$ and denote by X_σ the indicator of the event that all the edges $(\sigma(i), \sigma(i+1))$ appear in T with this orientation. Since the orientation of different edges is chosen independently,

$$\mathbb{E}(X_\sigma) = \mathbb{P}((\sigma(i), \sigma(i+1)) \in T \text{ for } i \in \{1, \cdots, n-1\}) = \frac{1}{2^{n-1}}.$$

To each of these permutations corresponds a distinct Hamiltonian path on the tournament which visits in the order the vertices $\sigma(1), \sigma(2), \cdots, \sigma(n)$. Since there are $n!$ possible permutations, the expected number of Hamiltonian paths is at least $\frac{n!}{2^{n-1}}$. As a result, there exists a (deterministic!) tournament which has at least this number of Hamiltonian paths.

Problem 4. Let V be the set of vertices of $G(n,p)$ and let X be the random variable that counts the number of triangles in $G(n,p)$.

a) Using the same technique seen in class (write X as the sum of indicator functions and use linearity of expectation), we have that

$$\mathbb{E}(X) = \sum_{\substack{S \subseteq V \\ |S|=3}} \mathbb{E}(\mathbb{1}_{G(S)=K_3}) = \sum_{\substack{S \subseteq V \\ |S|=3}} \mathbb{P}(G(S) = K_3) = \binom{n}{3} p^3,$$

where $G(S)$ is the subgraph with set of vertices S and K_3 is the complete graph with 3 vertices. Indeed, for a given triplet of vertices, the probability that they form a triangle is p^3 and there are $\binom{n}{3}$ of such triplets.

Pick $p = o(1/n)$. Then $\mathbb{E}(X)$ goes to 0 as n becomes large and we conclude using the first order method:

$$\mathbb{P}(G(n,p) \text{ contains a triangle}) = \mathbb{P}(X > 0) \leq \mathbb{E}(X) \to 0.$$

b) Let us first compute $\mathbb{E}(X^2)$:

$$\mathbb{E}(X^2) = \sum_{\substack{S,T \subseteq V \\ |S|=|T|=3}} \mathbb{E}(\mathbb{1}_{G(S)=K_3} \mathbb{1}_{G(T)=K_3}) = \sum_{\substack{S,T \subseteq V \\ |S|=|T|=3}} \mathbb{P}(G(S) = K_3, G(T) = K_3).$$

Take S and T so that they have at most 1 vertex in common. There are $\binom{n}{3}\binom{n-3}{3} + \binom{n}{3}\binom{3}{1}\binom{n-3}{2}$ distinct ways of doing this and the probability that $G(S)$ and $G(T)$ are both triangles is p^6.

Take S and T so that they have exactly 2 vertices in common. There are $\binom{n}{3}\binom{3}{2}\binom{n-3}{1}$ distinct ways of doing this and the probability that $G(S)$ and $G(T)$ are both triangles is p^5.

Take S and T so that they have exactly 3 vertices in common. There are $\binom{n}{3}$ distinct ways of doing this and the probability that $G(S)$ and $G(T)$ are both triangles is p^3.

As a result,

$$\sum_{\substack{S,T \subseteq V \\ |S|=|T|=3}} \mathbb{P}(G(S) = K_3, G(T) = K_3) = \binom{n}{3}\left(\binom{n-3}{3} + \binom{3}{1}\binom{n-3}{2}\right) p^6$$
$$+ \binom{n}{3}\binom{3}{2}\binom{n-3}{1} p^5 + \binom{n}{3} p^3.$$

Let us now compute the variance $\operatorname{var}(X)$:

$$\operatorname{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \binom{n}{3} p^3 \left(-(3n-8)p^3 + 3(n-3)p^2 + 1\right).$$

Pick p s.t. $1/n = o(p)$, namely, $\lim_{n\to\infty} pn = \infty$. Then, using the second order method, we have that

$$\mathbb{P}(G(n,p) \text{ does not contain a triangle}) = \mathbb{P}(X = 0) \leq \frac{\operatorname{var}(X)}{\mathbb{E}(X)^2}$$
$$= \frac{-(3n-8)p^3 + 3(n-3)p^2 + 1}{\binom{n}{3} p^3} = O\left(\frac{1}{n^2} + \frac{1}{n^2 p} + \frac{1}{(np)^3}\right) \to 0.$$