Graph Theory Applications — EPFL, Spring 2014

Problem Set 12

Date: 22.05.2014 — Not graded

Problem 1. Given a permutation σ on $\{1, 2, \cdots, n\}$, we say that $i \in \{1, 2, \cdots, n\}$ is a fixed point of the permutation if $\sigma(i) = i$. Pick a permutation uniformly at random from the set of all permutations on $\{1, 2, \cdots, n\}$. What is the expected number of fixed points?

Problem 2. Take a circle, color $5/6$ of it black and the rest red (not necessarily a contiguous part). We say that an inscribed regular pentagon is black if all its vertices are on the black part of the circle. Prove that there exists a black pentagon.

*H*int. Pick an inscribed regular pentagon at random. What is the probability that it is black?

Problem 3. Recall that in Problem 4 of Problem Set 2 we proved that every tournament has a Hamiltonian path. The aim of this exercise is to show that there exists a tournament with *a lot* of Hamiltonian paths. Show that there is a tournament on n vertices that has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

Hint. Consider a random tournament T where the orientations of the edges are chosen independently and uniformly. Write the expected number of Hamiltonian paths as the sum of expectations of indicator functions.

Problem 4. Let $G(n, p)$ be a graph with n vertices s.t. for any pair of nodes the edge connecting them is included in the graph with probability p independently from every other edge. In this exercise, we are interested in finding how to choose p as a function of n (for n large) so that with high probability $G(n, p)$ contains a triangle (clique of size 3).

a) Suppose that p decays faster than $1/n$, i.e. $p(n) = o(1/n)$. Show that the probability that $G(n, p)$ contains a triangle tends to 0.

b) Suppose that p decays slower than $1/n$, i.e. $1/n = o(p(n))$. Show that the probability that $G(n, p)$ contains a triangle tends to 1.

Note. If conditions a) and b) hold, we say that $1/n$ is a threshold function for the property "containing a triangle".