Solution to Problem Set 11

Date: 13.05.2014 Not graded

Problem 1. Let G be a bipartite graph with bipartition (U, V). Add a source s and connect all the vertices in U to s with capacity 1 edges. Add a sink t and connect all the vertices of V to t with capacity 1 edges. Set the capacity of all the edges in G to infinity. It is clear that any flow of value k in this new graph corresponds to a matching of size k. Indeed, a flow of value k connects k vertices of U to k vertices of V. Hence, the Ford-Fulkerson algorithm which finds the maximum flow of this network can be used to find the maximum matching of the underlying bipartite graph.

For the cuts, any finite value cut has to avoid the infinite capacity edges. Hence, every cut of size k corresponds to a vertex cover of size k. Finally, from the max-flow min-cut theorem (which states that the value of the maximum flow equals the capacity of a minimum cut) we can deduce that the sizes of the maximum matching and minimum vertex cover are equal in a bipartite graph.

Problem 2. Start with variable node 1 and its neighbors. If the variable node has less than l distinct neighbors then its expansion is at most $1 - 1/l$ and we are done. If variable node 1 has l neighbors then expand out further one layer. If this subgraph is not a tree then at least one variable leaf node must have two connections to the interior check nodes. Take this node and the root variable node. Together they have at most $l + (l - 2)$ neighbors, leading again to an expansion of at most $1 - 1/l$. Finally, if this subgraph is a tree, then we have $1 + l(r - 1)$ variable nodes and $l + l(r - 1)(l - 1)$ check nodes, which gives the expansion indicated in the problem statement.

The same type of arguments apply if we look at larger neighborhoods starting at a particular variable node. For larger and larger neighborhoods we see that the maximum achievable expansion tends to $1 - 1/l$.

Problem 3. For the last part of the question (the criterion which guarantees that any error pattern of weight two can be corrected), assume that no two variable nodes share more than one check in common and that the graph contains no cycle of length six or smaller. Let us also assume that there are no repeated edges, i.e., each variable node is connected to 3 distinct check nodes.

Let u and v be the variable nodes whose value is flipped during transmission.

Assume at first that they have no check in common. Then, each is connected to three unsatisfied check nodes and no other node is connected to more than two. In this case it is easy to see that we will decode correctly.

Suppose now that they do have a check in common (which will be satisfied). Hence, each of them is also connected to two unsatisfied check nodes and these two couples contains four different check nodes. However, in this case it is easy to see that no other variable node can be connected to more than one unsatisfied check node. Indeed, if this were the case, the graph would contain a cycle of length six. Therefore, also in this case we will decode correctly.

Problem 4. The first two points are trivial and true basically by definition. Only the last point needs a proof. Assume first that $n = 1$. In this case, if $x = z$ then the right-hand side is 0 and the left-hand side is either 0 or 2 and hence the inequality is correct. If $x \neq z$ then the right-hand side is equal to 1 and so is the left-hand side, irrespective of the value of y, and so again the inequality is fulfilled. Take now $n > 1$. Then, for each of the components of x, y, and z the triangle inequality holds via the previous argument. By summing up all these inequalities, the claim follows.