Problem Set 1

Date: 21.02.2014 Not graded

Problem 1. Suppose that n people are attending a party and there are some handshakes between different people in the party. Show that there are at least two persons who have shaken hands with the same number of people.

Hint. Pigeonhole.

Problem 2. Balls of 8 different colors are placed in 6 jars. There are 20 balls of each color. Show that there must be a jar containing two pairs from two different colors of balls (for example, there is a jar containing at least two blue and at least two green balls).

Problem 3. Prove that for any $n \in \mathbb{N}$,

$$\sum_{\substack{i=0 \\ i\ even}}^{n} \binom{n}{i} = 2^{n-1},$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ denotes the binomial coefficient.

Hint. There is a lot of ways in which one can prove the claim. One way is to show first that $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$ *with the binomial theorem.*

Problem 4. Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be a nonincreasing sequence of non-negative integers. We say that $\mathbf{d}$ is *graphic* if there exists a simple graph with degree sequence $\mathbf{d}$. Recall that in class we discussed that for the sequence $\mathbf{d}$ to be *graphic* we need that

$$\begin{aligned} &\sum_{i=1}^{n} d_i \text{ is even}, \\ &\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \qquad 1 \leq k \leq n. \end{aligned} \tag{1}$$

The aim of this exercise is to show an algorithm to construct a graph with degree sequence $\mathbf{d}$, if such a graph exists.

1. Suppose that $\mathbf{d}$ is graphic, i.e., there exists a graph G which has degree sequence $\mathbf{d}$. Show that there exists a graph G^* s.t. the vertex with degree d_1 is connected to the vertices with degrees $d_2, d_3, \cdots, d_{d_1+1}$.

2. Let $\mathbf{d}' = (d_2 - 1, d_3 - 1, ..., d_{1+d_1} - 1, d_{2+d_1}, \cdots, d_n)$. Use the previous result to show that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

3. Now, what could be an algorithm which constructs a graph with degree sequence $\mathbf{d}$ (if such a graph exists)?

Note that with the procedure outlined in this exercise we can prove that the conditions (1) are also *sufficient* for the sequence $\mathbf{d}$ to be graphic.

For the next two problem we are going to need some definitions. Let $d(u, v)$ denotes the graph distance between the vertices u and v ($u, v \in V(G)$), which is the minimum length of the paths connecting them, i.e., the length of the shortest path.

Let $\text{diam}(G)$ denote the diameter of G, which is the longest shortest path between any two vertices of the graphs, i.e.,

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

Let $\text{ecc}(v)$ denote the eccentricity of the vertex $v \in V(G)$, which is defined as the maximum graph distance between v and any other vertex $u \in V(G)$, i.e.,

$$\text{ecc}(v) = \max_{u \in V(G)} d(u, v).$$

Let $\text{rad}(G)$ denote the radius of the graph G, which is defined as the minimum graph eccentricity of any graph vertex in G, i.e.,

$$\text{rad}(G) = \min_{v \in V(G)} \text{ecc}(v).$$

Let $N(v)$ denote the neighborhood of a vertex $v \in V(G)$, which is defined as the set of all vertices adjacent to v including v itself. By extension, the neighborhood $N(S)$ of a set $S \subseteq V(G)$ of vertices is defined as the union of the neighborhoods of the vertices $v \in S$, i.e.,

$$N(S) = \bigcup_{v \in S} N(v).$$

Problem 5. Show that for every connected graph G the following inequalities hold:

$$\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G).$$

Find a graph where $\text{rad}(G) = \text{diam}(G)$, and a graph where $2\text{rad}(G) = \text{diam}(G)$.

Problem 6. If the maximum degree of a connected bipartite graph G is $\Delta(G)$, prove that the maximum number of vertices in it is

$$|V(G)| \leq 2\frac{(\Delta(G) - 1)^{\text{diam}(G)} - 1}{\Delta(G) - 2}.$$