Graph Theory Applications EPFL, Spring 2014

Final Exam

Date: 27.06.2014

Rules:

- This exam is closed book. No electronic items are allowed. You are only allowed to have two handwritten single-sided A4 pages of notes. Place all your personal items on the floor. Leave only a pen and your ID on the desk. If you need extra scratch paper, please ask for it by raising your hand.
- Please do not cheat. We will be forced to report any such occurrence to the president of EPFL. This is not how you want to meet him. :-(
- The exam starts at 8:15 and lasts till 11:00 in INM10.
- If a question is not completely clear to you, don't waste time and ask us for clarification right away.
- It is not necessarily expected that you solve all problems. Don't get stuck. Start with the problem which seems the easiest to you and try to collect as many points as you can.

Name : ....

Sciper : ....

Problem 1 ... / 20

Problem 2 ... / 20

Problem 3 ... / 20

Problem 4 ... / 20

Problem 5 ... / 20

Problem 6 – Bonus ... / 10

TOTAL ... / 100+10

Problem 1. [*20pts*] Let $G = (V, E)$ be a graph with m edges. Prove that there exists a subset of E, call it F, of cardinality at least $m/2$ so that the induced subgraph $G(F)$ is bipartite.

Hint. Probabilistic method.

Problem 2. [*20pts*] Consider an (l, r)-regular bipartite graph G with $2 \leq l \leq r$ and n variable nodes representing an error correcting code. Assume that the graph is an expander with expansion exceeding $1/2$ for all sets of variable nodes of size up to αn, $0 < \alpha \leq 1$. In class we showed that the corresponding code has a minimum distance of at least αn, i.e., any non-zero codeword of this code has Hamming weight at least αn.

Let us define a *stopping set*, call it $\mathcal{S}$, as a subset of the set of variables so that in the induced subgraph $G(\mathcal{S})$ there is no check node of degree 1. To compare, one can think of a codeword as a subset of the set of variable nodes, call it $\mathcal{S}$, so that in the induced subgraph $G(\mathcal{S})$ all check nodes have even degree.

Prove that in such a graph any non-zero stopping set must have size at least αn, i.e., there are no small stopping sets.

Problem 3. [*20pts*] Suppose that several families want to go to a picnic by car sharing. Unfortunately, the members of each family constantly quarrel among themselves! To ensure a relaxing trip, the coordinator wants to make sure that no two members of the same family travel in a car together. Given a set of n families with $F_1, F_2, \ldots, F_n$ members and a set of m cars with capacity $C_1, C_2, \ldots, C_m$, write this problem as a graph problem and suggest an efficient algorithm to solve it.

Problem 4. [*20pts*] Take an 8×8 chessboard and remove from it the two squares with coordinates $(1, 1)$ and $(8, 8)$ (these are two squares opposite to each other along a diagonal).

Show that it is not possible to cover the remaining area consisting of 62 squares with dominoes of size 1×2 (the dominoes can be put horizontally or vertically).

Hint. Phrase this as a matching problem and show that no appropriate matching exists.

Problem 5. [*20pts*] Consider the complete graph on 6 vertices. Show that, for any coloring of the edges with blue and red, there exists always either a red or a blue triangle.

Problem 6. *[Bonus – 10points] Déjà vu.* Consider the graph $G(V, E)$ with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(Note that this graph one vertex has a self loop.)

This matrix has a maximum eigenvalue of $\lambda = 1 + \sqrt{2}$ and an associated eigenvector $x^T = (1, 1/\sqrt{2}, 1/\sqrt{2})$.

Give a good upper and lower bound on the number of walks of length n from vertex 1 back to vertex 1.

Note: *Good* here means that the ratio of the two bounds is $\Theta(1)$, i.e., it stays constant as n grows large.