Exercises november 9, 2007. Quantum information theory and computation

Exercise 1. Mixtures

a) Show that the two mixtures $\{|0\rangle, \frac{1}{2}; |1\rangle, \frac{1}{2}\}$ and $\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle), \frac{1}{2}\}$ have the same density matrix.

b) Consider the mixture $\{|0\rangle, \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}\}$. Give the spectral decomposition of the density matrix.

Exercise 2. Intermediate state in teleportation

a) In the teleportation protocol, just after Alice's measurement, what is the density matrix that Bob should use to describe his state assuming he has no information on Alice's measurement ?

b) Assuming that one can associate an entropy to Bob's density matrix what should this be ?

Exercise 3. Reduced density matrix

a) Take the first GHZ state for three Qbits $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Compute the reduced density matrices ρ_{AB} and ρ_C.

b) Take the state $|\Phi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ used in teleportation. Compute Alice's and Bob's reduced density matrices.

c) Check that the Schmidt decomposition theorem holds in each of the above cases.

Exercise 4. Schmidt decomposition theorem

Consider the pure N Qbit state,

$$|\Psi\rangle = \frac{1}{2^{N/2}} \sum_{b_1...b_N \in \{0,1\}^N} |b_1...b_N\rangle$$

a) Compute the density matrix of the first Qbit. Show that it has non degenerate eigenvalues 0 and 1.

b) Compute the reduced density matrix of the set of bits $(2...N)$. Show that this $2^{N-1} \times 2^{N-1}$ matrix has a non degenerate eigenvalue 1 and an eigenvalue 0 with degeneracy $2^{N-1} - 1$.

c) Check explicitly that the Schmidt decomposition theorem holds.

Exercise 5. Purification

Consider the pure the mixed state $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$. Give two purifications of this state: one which is entangled and one which is a tensor product.