Exercises october 26, 2007. Quantum information theory and computation

Exercise 1. Bell inequality for a non-maximally entangled state.

Calculate the QM prediction for the CHSH quantity (we called it X in the lecture on Bell's inequality) when the EPR pair is produced in the state

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1-\alpha^2)^{1/2}|11\rangle$$

Repeat the calculations done in the notes to show that the maximal value of X is $2[1+4\alpha^2(1-\alpha^2)]^{1/2}$. In this sense we can say that $\alpha = \frac{1}{\sqrt{2}}$ corresponds to a maximally entangled state.

Exercise 2. Tsirelson inequality and maximal violation of Bell's inequality

The purpose of the exercise is to show that the set up described in the course yields the maximum possible violation of the Bell inequality.

The three 2×2 matrices X, Y, Z are called Pauli matrices. In the Dirac notation they are $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$. In physics the standard notation for these matrices is σ_x, σ_y and σ_z.

It is often convenient to introduce the "vector" $\sigma = (X, Y, Z)$. For electrons this has the physical meaning of the "spin of the electron". For photons it simply corresponds to three different polarisation observables: linear (say 45 degrees), circular, linear (say 0 degree).

a) Check the commutation relations $[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$.

b) Let $Q = \mathbf{q} \cdot \sigma$ and $R = \mathbf{r} \cdot \sigma$. Check $[Q, R] = 2i(\mathbf{q} \times \mathbf{r}) \cdot \sigma$

c) Let also $S = \mathbf{q} \cdot \sigma$ and $T = \mathbf{t} \cdot \sigma$. Prove the identity

$$R \otimes S + R \otimes S + R \otimes T - Q \otimes T = 4I + [Q, R] \otimes [S, T]$$

and deduce that for any state $|\psi\rangle$ of $\mathbf{C}^2 \otimes \mathbf{C}^2$ we have the inequality

$$\langle\psi| R \otimes S + R \otimes S + R \otimes T - Q \otimes T |\psi\rangle \leq 2\sqrt{2}$$

d) What are $|\psi\rangle$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ in the experimental setup of described in the course on the violation of Bell's inequality? What is the general significance of the above inequality ?

Exercise 3. GHZ states and "local hidden variable theories"

The goal of this exercise is to discuss a thought experiment that proves that QM results cannot be replaced by local hidden variable theories, in an even stronger sense than the CHSH inequality violation. In the latter one both party has to do many measurements and then look at the correlation between outcomes. In what follows we will see that only one measurement is made by three parties and they can decide that QM wins just by looking at their results (they still have to meet or communicate).

Consider a GHZ state of three spins $|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle_{ABC} - |\downarrow\downarrow\downarrow\rangle_{ABC})$ where A, B, C are distant locations (which do not communicate). Consider the three observables X, Y, Z represented by the three Pauli matrices (actually we will not use Z so forget about it).

a) Show that $|GHZ\rangle$ is an eigenstate of the operators $Y_A \otimes Y_B \otimes X_C$, $Y_A \otimes X_B \otimes Y_C$, $X_A \otimes Y_B \otimes Y_C$ with eigenvalue 1. Furthermore show that $|GHZ\rangle$ is an eigenstate of $X_A \otimes X_B \otimes X_C$ with eigenvalue -1.

b) Now imagine Alice, Bob and Charlie in their labs at locations A, B and C measure the observables X and Y on their respective particles. They do the four experiments (each time on a new GHZ state):

- experiment one: Alice measures Y, Bob Y and Charlie X.
- experiment two: Alice measures Y, Bob X and Charlie Y.
- experiment three: Alice measures X, Bob Y and Charlie Y.
- experiment four: Alice measures X, Bob X and Charlie X.

Give the resulting states and the associated probability after each experiment according to QM.

c) Suppose now that the outcome of any measurement can be described by a local hidden variable theory. In other words suppose that Alice, Bob and Charlie have some way of computing the outcome of their experiments by some functions $F_A(W, \Lambda)$, $F_B(W, \Lambda)$, $F_C(W, \Lambda)$ where the first variable W is the measurement basis (or apparatus) used i.e $W = X, Y$ and the second variable Λ is the "hidden variable" of the theory (e.g state of the rest of the universe). Show that this setting is not compatible with the QM results of the four previous experiments.

Hint: there is no big calculation, you only have to multiply plus and minus ones ! When the spin is $\uparrow$ record a $+1$ for $F_{A,B,C}(W, \Lambda)$ and when it is $\downarrow$ record a -1 for $F_{A,B,C}(W, \Lambda)$.

Exercise 4. Entanglement swapping

Let O, A,A', B and B' be located at coordinates 0, $-L$, $-\frac{L}{2}$, L and $\frac{L}{2}$ respectively. We suppose that two EPR pairs are produced at A' and B'. For each pair the entangled particles are then propagated to A and O and to B and O. Thus we have an entangled Bell state between A and O and another entangled Bell state between B and O. If the state of the four particles (or four Qbits) is

$$\frac{1}{\sqrt{2}}(|00\rangle_{AO} + |11\rangle_{AO}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{OB} + |11\rangle_{OB})$$

explain what happens if we make a measurement in the Bell basis of the two Qbits located at O.

Now consider three closeby locations A, B, C (for example three points in your lab) and three distant locations A', B', C'. Suppose we have created three entangled pairs between AA', BB', CC' in the state

$$\frac{1}{\sqrt{2}}(|00\rangle_{AA'} + |11\rangle_{AA'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{BB'} + |11\rangle_{BB'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{CC'} + |11\rangle_{CC'})$$

What happens if we do a measurement in the GHZ basis of the three particles at A, B, C ?

Hint: The states of the 8 dimensional basis of fully entangled GHZ states are $\frac{1}{\sqrt{2}}(|000\rangle_{ABC}+|111\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|000\rangle_{ABC}-|111\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|001\rangle_{ABC}+|110\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|001\rangle_{ABC}-|110\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|010\rangle_{ABC}+|101\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|010\rangle_{ABC}-|101\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|100\rangle_{ABC} + |011\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|100\rangle_{ABC} - |011\rangle_{ABC})$.