Exercises December 7 and 14, 2007. Quantum information theory and computation

Exercise 1. Quantum Toffoli gate

Check that the quantum Toffoli gate can be obtained from the circuit in the lecture notes made out of the set $\{T, S, H, CNOT\}$

Exercise 2. Complexity of Deutsch model

Suppose that we approximate single bit gates $\mathbf{C^2} \to \mathbf{C}^2$ to a precision ϵ (lemma 1). Then an arbitrary "two level unitary gate" acting on N Qbits $\mathbf{C}^{2^N} \to \mathbf{C}^{2^N}$ acting non trivialy on coordinates i and j and as the identity on all other coordinates is approximated to a presicion ϵ (see course). In other words given a two level unitary gate $U^{(ij)}$ we approximate it by $V^{(ij)}$ such that $||U^{(ij)} - V^{(ij)}|| \leq \epsilon$. Show that if an arbitrary unitary acting on N Qbits (this means it is a $2^N \times 2^N$ matrix) has a decomposition (lemma 3)

$$U = U^{(i_1 j_1)} ... U^{(i_K j_K)}$$

the error accumulate for U is of the order $O(\epsilon K)$.

The consequence is that if we want an overall precision of δ for the circuit of U we need a presicion of $\frac{\delta}{K}$ for each two level unitary. It can be shown that this can be done with $O(\ln(\frac{\delta}{K}))$ gates. So the total number of gates will increase like $O(K \ln(\frac{\delta}{K}))$.

Recall that there exist unitary U for which K is necessarily exponentialy big in N.

hint: The norm of a matrix is $||A|| = \sup_{|\psi\rangle} \frac{||A\psi\rangle||}{|||\psi\rangle||}$. In the above one just uses the triangle inequality.

Exercise 3. Deutsch-Josza problem

Check the calculations of the course leading to the formula for the probability of outcome equal to $(a_1, ..., a_N) = (0, ..., 0)$,

$$Prob(0, ..., 0) = \frac{1}{2^{2N}} \Big| \sum_{b_1,...,b_N} (-1)^{f(b_1,...,b_N)} \Big|^2$$

Exercise 4. Unitarity of QFT

Prove that the Quantum Fourier Transform

$$|x\rangle \to \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |y\rangle$$

is unitary.

Exercise 5. QFT on 3 Qbits

Using the Hadamard and *phase gates* S and T of lecture 10, give the QFT circuit for 3 Qbits (i.e $N = 0, ..., 7$) and write down explicitely the 8×8 unitary matrix.

Exercise 6. Tensor product decomposition of QFT

The binary representation of $x \in \{0, ..., N-1\}$ for $N = 2^n$ is

$$x = x_{n-1}.2^{n-1} + ... + x_2.2^2 + x_1.2^1 + x_0.2^0, \qquad x_i \in \{0, 1\}$$

Prove that the QFT of $|x\rangle$ is equal to the tensor product

$$(|0\rangle + e^{\pi i \frac{x}{2^0}} |1\rangle) \otimes (|0\rangle + e^{\pi i \frac{x}{2^1}} |1\rangle) \otimes ... \otimes (|0\rangle + e^{\pi i \frac{x}{2^{n-1}}} |1\rangle)$$

Hint: In the QFT of $|x\rangle$ represent $|y\rangle = |y', y_0\rangle$ where $y_0 = 0$ or $y_0 = 1$ and inspect each contribution $y_0 = 0$ or 1.