Exercises november 23, 2007. Quantum information theory and computation

Exercise 1. Holevo bound

Consider a source with preparation $\{|0\rangle, \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\}$. Investigate if the Holevo bound on the mutual information between the preparation X and the measurement results Y

$$\max_{all\ measurements} I(X;Y) \leq \chi(\rho)$$

is saturated.

Exercise 2. Source coding of mixed states

We consider a source of mixed states ρ_x occuring each with probabilities p_x. Messages are N letter strings of the form $\rho_{x_1} \otimes .... \otimes \rho_{x_2}$ and have a probability $p_{x_1}...p_{x_N}$. In this exercise we want to give some support to the conjecture that the achievable rate of compression for a source of mixed states is equal to the Holevo quantity

$$\chi(\rho) = S(\rho) - \sum_x p_x S(\rho_x), \qquad \rho = \sum_x p_x \rho_x$$

Note that in the case of a source of pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ the Holevo quantity $\chi(\rho)$ reduce to $S(\rho)$ which is the optimal achievable rate given by Schumacher's theorem.

a) Take a source constituted of the unique letter ρ_0 occuring with probability $p_0 = 1$. How many bits are needed to compress this source ? What is the value of $\chi(\rho)$? Is this consistent ?

b) Now consider a source of mixed mutually orthogonal states. Two mixed states are said to be mutually orthogonal if

$$Tr \rho_x \rho_y = 0, \qquad x \neq y$$

Consider purifications $|\Psi_x\rangle$ of ρ_x. Show that these satisfy

$$\langle\Psi_x|\Psi_y\rangle = 0, \qquad x \neq y$$

What would be an encoding scheme achieving a compression rate of $H(X) = -\sum_x p_x \log p_x$? Why would this rate be optimal ? Check that in the present case we have

$$H(X) = \chi(\rho)$$

hint: no big calculations