Discrete Structures | EPFL, Fall 2013

Problem Set 13

Date: 13.12.2013 | Not graded

Problem 1.

i) Let us toss 3 times a fair coin and let us denote by A the event that the outcomes are not all equal and by B the event that at most 1 Heads is obtained. Prove or disprove that the events A and B are independent.

ii) How many times should we throw a fair die (with equiprobable faces numbered from 1 to 6) so that the probability of getting at least a 6 is greater than 99%?

iii) A red and a green die are rolled. What is the probability of getting a sum of 6, given that the number on the red die is even? What is the probability of getting a sum of 6, given that the number on the red die is odd?

Problem 2. You are given a mysterious box with a button and 3 lights, each of which can be on or off at random. You play around for a while with the box, and you find that after pushing the button, a random configuration of lights appears. However, only some configurations are possible, while others never appear. Namely, the configurations that appear are exactly those where an odd number of lights are on, and these configurations appear with equal probability.

i. Show that for each of the lights, the probability that it goes on is 1/2.

ii. Show that any two lights are independently on and off, i.e. the lights are pairwise independent.

iii. Show that the three lights are not independent.

Problem 3. You are playing the following game of chance. In each turn you place a bet of x\$, after which a fair coin is flipped. If it produces *tails*, you receive your bet x\$ plus an extra x\$; otherwise you lose the x\$ from the bet. You have the following strategy: in the first turn you bet 1\$; then every time you lose, you double your bet. If you cannot afford to double your bet, you leave the game. In case you win, you also stop the game and walk away with the money.

a) If you have at disposal 255\$, what is the probability that you walk away broke? What is the expected value of your total gains? (In case of losses, gains are negative)

b) Do the same for the case where you have an infinite supply of money.

Problem 4.

i) During an oral exam, an Evil Teaching Assistant (ETA) prepares 2 very hard questions, and 1 very easy question and puts them in 3 identical boxes. The student is then allowed to pick one box, without looking inside. In order to maximize his chances of getting a perfect grade, the student would like to pick the box containing the very easy question. What is the probability of picking the very easy question?

ii) The ETA announces to the student that he will remove one of the hard questions. He opens one of the two remaining boxes, and proceeds to read the question to the student, confirming that it was indeed a very hard question. The ETA then offers the student the choice to exchange the box he picked at the beginning for the remaining, unpicked box. Would you keep the original box or exchange it? What is the probability that the very easy question is in the unpicked box?

Problem 5. You are conducting a poll on an upcoming referendum. There are $N = 10^8$ voters, out of which an unknown number N' will vote *yes* and $N - N'$ will vote *no* (there are no abstentions). We are interested in estimating the ratio $p = N'/N$ of people who vote *yes*. To perform the estimation, you will choose K voters independently and uniformly at random (this means you might pick the same person more times) and take their answers as representative of the whole. Thus, if among the K people you ask, K' declare that they will vote *yes*, you declare K'/K as your estimation of p. Moreover, assume that people do not lie.

a) Let A_j be 1 if the j-th person interrogated votes *yes* and 0 if they vote *no*. What are the probabilities for A_j to take its possible values? What are the expectation and variance of A_j?

b) Write your estimate $X = K'/K$ as the the mean of the A_j's, $X = \frac{1}{K}\sum_{j=1}^{K} A_j$. Compute the expectation of X and its variance. Hint: the variance of a sum of independent random variables is the sum of the individual variances.

c) Given some parameter ϵ, use the Chebyshev inequality to upper bound the probability that the absolute difference $|X - p|$ (i.e. the *error*) exceeds ϵ. A numerical calculation: suppose $p = 1/2$ and $\epsilon = 0.05$.

Problem 6. There is a poll in which the two candidates Alice and Bob obtain a votes and b votes, respectively. Let $a > b$, i.e., Alice wins the election. Suppose that at each moment of the counting of the votes, the remaining electoral ballots are equally likely to be extracted. Denote by p the probability that during the whole counting Alice always has (strictly) more preferences than Bob. The aim of this exercise is to evaluate p.

a) Model a generic counting of the votes by $x = (x_1, x_2, \cdots, x_N)$, where $N = a+b$, $x_i = 1$ if the i-th vote is for Alice, and $x_i = -1$ if the i-th vote is for Bob ($i \in \{1, 2, \cdots, N\}$). Let $s_n = \sum_{i=0}^{n} x_i$, with $n \in \{1, 2, \cdots, N\}$. The sample space is defined as $\Omega = \{x \in \{-1, +1\}^N | s_N = a - b\}$. Convince yourself that the sets A, B, and C defined below constitute a partition of Ω:

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\begin{aligned}
A &= \{x \in \{-1, +1\}^N \mid s_N = a - b,\ \forall n \geq 1\ s_n \neq 0\}, \\
B &= \{x \in \{-1, +1\}^N \mid s_N = a - b,\ x_1 = 1,\ \exists n \geq 1 \text{ s.t. } s_n = 0\}, \\
C &= \{x \in \{-1, +1\}^N \mid s_N = a - b,\ x_1 = -1\}.
\end{aligned}
$$

b) Observe that the sentence "*Suppose that at each moment of the counting of the votes, the remaining electoral ballots are equally likely to be extracted*" implies that all the elements of Ω are equally likely. Compute $|\Omega|$.

c) Prove that $|B| = |C|$ by showing a bijection between these two sets.

d) Compute $|C|$.

e) Using the results of the previous points, evaluate p.