Discrete Structures EPFL, Fall 2013

Problem Set 7

Date: 1.11.2013 Not graded

Problem 1.

a) Recall that $n!$ is defined as the product of all natural numbers between 1 and n, i.e., $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$. In how many 0's does 104! end (in the usual, decimal notation)?

b) Compute the remainder of the division of $25^{3702298}$ by 53.

c) Prove or disprove: A positive integer congruent to 1 modulo 4 cannot have a prime factor congruent to 3 modulo 4.

d) Prove or disprove: If p and q are primes greater than 2, then $p \cdot q + 1$ is never prime.

e) Prove or disprove: There exist two consecutives primes, each of them being greater than 2.

Problem 2. Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

Hint 1: Write n as $2^k + j$ where $k \in \mathbb{N}$ and $j \in \{0, \ldots, 2^k - 1\}$. Why is this representation unique?
Hint 2: Observe that $2^k + j \leq 2 \cdot 2^k$ for any $k \in \mathbb{N}$ and any $j \in \{0, \ldots, 2^k - 1\}$.

Problem 3. In this problem we will show, by steps, that also $\displaystyle\sum_{\substack{p \geq 1 \\ p \text{ prime}}} \frac{1}{p}$ diverges.

Let p_k be the k^{th} largest prime, and let the set $\mathcal{A}_n$ be defined by

$$\mathcal{A}_n = \{p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} : 0 \leq i_1, i_2, \ldots, i_n \leq n\}.$$

It is very important that you work the details of each step!

a) Convince yourself, using the unique factorization theorem, that

$$\left(1 + \frac{1}{p_1} + \ldots + \frac{1}{p_1^n}\right)\left(1 + \frac{1}{p_2} + \ldots + \frac{1}{p_2^n}\right) \ldots \left(1 + \frac{1}{p_n} + \ldots + \frac{1}{p_n^n}\right) = \sum_{m \in \mathcal{A}_n} \frac{1}{m}. \quad (1)$$

b) Use the sum of the geometric series to show that the left hand side of the equation above is upper bounded by

$$\left(1 + \frac{1}{p_1 - 1}\right)\left(1 + \frac{1}{p_2 - 1}\right) \ldots \left(1 + \frac{1}{p_n - 1}\right).$$

c) Take the natural logarithm on both sides of (1) and, using b), show that

$$\sum_{j=1}^{n} \frac{1}{p_j - 1} \geq \ln \sum_{m \in \mathcal{A}_n} \frac{1}{m}.$$

d) After proving that $\{1, \ldots, n\} \subseteq \mathcal{A}_n$, use it to show that

$$1 + \sum_{j=1}^{n-1} \frac{1}{p_j} \geq \ln \sum_{m=1}^{n} \frac{1}{m}. \tag{2}$$

e) Conclude, using the statement of the previous problem, that $\sum_{j=1}^{n} \frac{1}{p_j}$ diverges when $n \to \infty$.

Problem 4. Use mathematical induction (in its original or strong form) to show the following:

a) For any $n > 1$, given the sets $A_1, \ldots, A_n$,

$$A_1 \cap (A_2 \cup \ldots \cup A_n) = (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \ \ldots \ \cup (A_1 \cap A_n).$$

b) For all $n > 0$, it holds that $13 \mid 17^n - 4^n$.

c) The Swiss National Bank goes insane and decides that from now on it will only mint coins of 5 francs and 9 francs. Show that you can pay any amount of francs strictly greater than 31 using such coins.

d) For all $n > 0$, it holds that $\sum_{i=0}^{n} (-2)^i = \frac{(-1)^n \cdot 2^{n+1} + 1}{3}$.

Problem 5. Find the error in the proof of the following "theorem".

Theorem. Let $\mathcal{A}$ be any set of n girls. Then, all the girls in $\mathcal{A}$ have the same hair color.

Proof. By induction.
Base step, $n = 1$: As $\mathcal{A}$ contains exactly one girl, the statement is trivially true.
Induction step: Assume that in any set of n girls, all elements have the same color.
Let us consider a set $\mathcal{A}$ of $n + 1$ girls and let us remove a girl g from it, obtaining a set $\mathcal{A}'$ consisting of n girls. Then by the induction hypothesis, all the girls in $\mathcal{A}'$ have the same hair color. Now, let us put back the girl g, and instead remove another random girl, obtaining a set $\mathcal{A}''$. As $\mathcal{A}''$ still consists of n girls, by the induction hypothesis, we can conclude that the girl g (which is inside $\mathcal{A}''$) has the same hair color as the others. Thus $\mathcal{A}$ consists of $n + 1$ girls with the same hair color. □