Discrete Structures EPFL, Fall 2013

Solution to Problem Set 13

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Problem 1.

i) The complement of A occurs when 3 Heads or 3 Tails are obtained, which happens with probability $\frac{1}{4}$. Hence, $\mathbb{P}(A) = 1 - \frac{1}{4} = \frac{3}{4}$. The event B occurs when exactly 1 Heads is obtained or no Heads are obtained. Then, $\mathbb{P}(B) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$. The event $A \cap B$ occurs when exactly 1 Heads is obtained. Then, $\mathbb{P}(A \cap B) = \frac{1}{8} = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Hence, the events A and B are independent.

ii) Let n be the number of times the die is thrown and let $p(n)$ be the probability that we obtain at least a 6. Then, $p(n) = 1 - \left(\frac{5}{6}\right)^n$. Consequently, after some calculations, we have that

$$p(n) > \frac{99}{100} \Longleftrightarrow n > \frac{2}{\log_{10} 6 - \log_{10} 5}.$$

Since n has to be an integer, we need that $n \geq 26$.

iii) Let A be the event that the red die is even, B the event that the red die is odd, and C the event that the sum is 6. Observe that if the red die is 6, then no value of the green die can make the sum equal to 6. In all the other cases, there is a unique value of the green die which makes the sum 6. Therefore,

$$\mathbb{P}(C|A) = \frac{\mathbb{P}(C \cap A)}{\mathbb{P}(A)} = \frac{\frac{2}{36}}{\frac{1}{2}} = \frac{1}{9},$$
$$\mathbb{P}(C|B) = \frac{\mathbb{P}(C \cap B)}{\mathbb{P}(B)} = \frac{\frac{3}{36}}{\frac{1}{2}} = \frac{1}{6}.$$

Problem 2.

i. The equiprobable disjoint events are A_1, A_2, A_3, A_{123}, where the index shows which lights are on in that event. Since they are equiprobable, each has probability $1/4$. Consider one of the lights, without loss of generality let it be the first light. There are two disjoint events in which this light is on, A_1 and A_{123}, and so the probability of the first light being on is $1/2$. Obviously, since the lights are symmetric under renaming we can say the same thing about the others.

ii. Consider without loss of generality the first two lights. Let X_i be a random variable that is 1 when light i is on and 0 otherwise. Then

$$\begin{aligned}
\mathbb{P}(X_1 = 0 \wedge X_2 = 0) &= \mathbb{P}(A_3) = 1/4 = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0),\\
\mathbb{P}(X_1 = 0 \wedge X_2 = 1) &= \mathbb{P}(A_2) = 1/4 = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1),\\
\mathbb{P}(X_1 = 1 \wedge X_2 = 0) &= \mathbb{P}(A_1) = 1/4 = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0),\\
\mathbb{P}(X_1 = 1 \wedge X_2 = 1) &= \mathbb{P}(A_{123}) = 1/4 = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1),
\end{aligned}$$

which means that X_1 and X_2 are independent.

iii. The three lights are not independent, since

$$\mathbb{P}(X_1 = 1 \cap X_2 = 1 \cap X_3 = 1) = \frac{1}{4} \neq \mathbb{P}(X_1 = 1) \cdot \mathbb{P}(X_2 = 1) \cdot \mathbb{P}(X_3 = 1) = \frac{1}{8}.$$

This exercise shows that there can be events that are pairwise independent but still not independent!

Problem 3.

a) You walk away broke if you lose exactly 8 times in a row. This happens with probability $p_{\text{lose}} = 2^{-8}$, since the coin flips are independent. Note that whenever you win (w.p. $1 - p_{\text{lose}}$), you win exactly 1\$, so the expected value won is $p_{\text{lose}} \cdot (-255) + (1 - p_{\text{lose}}) \cdot 1 = 0$.

b) If you have an infinite supply of money, it is not possible to become broke, so $p_{\text{lose}} = 0$ and the expected value won is 1\$!

Problem 4.

i) The probability of picking the very easy question is clearly $\frac{1}{3}$.

ii) Suppose that the very easy question was picked initially. Then, it is in the student's box and this happens with probability $\frac{1}{3}$. Otherwise, if a very hard question was picked initially, then the easy question is in the remaining box, and this happens with probability $\frac{2}{3}$. Hence, you should exchange the original box, because the probability that the very easy question is in the unpicked box is equal to $\frac{2}{3}$. The reason why the probabilities change is that revealing a box with a very hard question gives the student some extra information about the other boxes.

Another way to see the same result is the following. Suppose without loss of generality that box 1 is chosen at the beginning. Then, the ETA opens box 2. Denote by A_i the event that the very easy question is in box i, with $i \in \{1, 2, 3\}$, and by B the event that the ETA opens box 2. The probability that the very easy question is in the unpicked box can be computed as follows,

$$\mathbb{P}(A_3|B) = \frac{\mathbb{P}(B|A_3)\mathbb{P}(A_3)}{\mathbb{P}(B|A_3)\mathbb{P}(A_3) + \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2)}.$$

Clearly, $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{1}{3}$. Since box 1 is chosen at the beginning, if the very easy question is in box 3, the ETA must open box 2. Then, $\mathbb{P}(B|A_3) = 1$. If the very easy question is in box 1, the ETA can open either box 2 or box 3. Hence, $\mathbb{P}(B|A_2) = \frac{1}{2}$. If the very easy question were in box 2, then the ETA would not have opened that box! Therefore, $\mathbb{P}(B|A_2) = 0$. As a result, we obtain $\mathbb{P}(A_3|B) = \frac{2}{3}$.

This problem is a form of the well-known Monty Hall problem.

Problem 5.

a) For each $j = 1, \dots, K$, with probability p the person chosen votes *yes* and with probability $1 - p$ she votes *no*. Thus $\mathbb{P}(A_j = 1) = p$ and $\mathbb{P}(A_j = 0) = 1 - p$. The expectation is given by $\mathbb{E}(A_j) = 1 \cdot p + 0 \cdot (1 - p) = p$. The variance is given by

$$\text{Var}(A_j) = \mathbb{E}(A_j^2) - (\mathbb{E}(A_j))^2 = p - p^2 = p(1 - p),$$

where we used the fact $A_j^2 = A_j$.

b) The expectation of X is given by $\mathbb{E}(X) = \frac{1}{K}\sum_{j=1}^{K}\mathbb{E}(A_j) = p$. Since the A_j are independent, the variance is also additive so we obtain

$$\mathrm{Var}(X) = \frac{1}{K^2}\mathrm{Var}(A_1 + \ldots + A_K) = \frac{1}{K^2}(\mathrm{Var}(A_1) + \ldots + \mathrm{Var}(A_K)) = \frac{p(1-p)}{K}.$$

c) We apply the Chebyshev inequality for X which reads for any $u \in \mathbb{R}_{>0}$

$$\mathbb{P}(|X - p| \geq u\sigma) \leq \frac{1}{u^2},$$

where σ is the standard deviation of X, defined by $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{p(1-p)}{K}}$. We wish to have ϵ instead of $u\sigma$ so we use $u = \epsilon/\sigma$ and obtain

$$\mathbb{P}(|X - p| \geq \epsilon) \leq \frac{p(1-p)}{K\epsilon^2}.$$

Setting $p = 1/2$, $K = 1000$ and $\epsilon = 0.05$, we get $\mathbb{P}(|X - p| \geq \epsilon) \leq \frac{1}{10}$. Note that this is only a bound! The actual value of the error probability is in fact somewhat smaller and can be approximated better using the Central Limit Theorem.

Problem 6.

a) The event A corresponds to the scrutinies in which the two candidates at no point in time have the same number of votes. The event B corresponds to the scrutinies which are not in A and s.t. the first vote is for Alice. The event C corresponds to the scrutinies which are not in A and s.t. the first vote is for Bob. Then, it is clear that the sets A, B, and C are disjoint and that their union is Ω.

b) The cardinality of Ω is given by the number of combination of a votes from the total of N votes (without replacement). In formulas,

$$|\Omega| = \binom{a+b}{a}.$$

c) Consider $f : B \to C$ defined as follows. Let $x = (x_1, \cdots, x_N) \in B$ and $y = (y_1, \cdots, y_N) \in C$ s.t. $y = f(x)$. Denote by k the smallest integer s.t. $\sum_{i=0}^{k} x_i = 0$. Notice that such a k must exist because $x \in B$. Then, take $y_i = -x_i$ for $i \leq k$ and $y_i = x_i$, otherwise. In words, the function f changes the first k votes, k being the point in which there is the first tie between Alice and Bob. It is easy to check that f is a bijection.

d) The cardinality of C is given by the number of combinations of a elements from a class of $a + b - 1$ elements without replacement. Indeed, the first vote is fixed and it is for Bob, so we need to count in how many ways we can extract a votes for Alice from a total of $a + b - 1$ votes (without replacement). Therefore,

$$|C| = \binom{a+b-1}{a}.$$

e) Remember that the elements of Ω are equally likely and that the scrutinies in which Alice has always strictly more preferences than Bob are the ones in A. Then, $p = \frac{|A|}{|\Omega|}$.

From point a), we have that A, B, and C form a partition of Ω. Then, $|\Omega| = |A| + |B| + |C|$. Using the fact that $|B| = |C| = \binom{a+b-1}{a}$, we obtain that

$$p = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |B| - |C|}{|\Omega|} = 1 - 2\frac{\binom{a+b-1}{a}}{\binom{a+b}{a}} = 1 - \frac{2b}{a+b} = \frac{a-b}{a+b}.$$