Solution to Graded Problem Set 12

Date: 6.12.2013 Due date: 13.12.2013, noon

Problem 1.

a) It is possible to write $F(x) = \sum_{k=1}^{5}\sum_{j=1}^{k} \frac{\beta_{k,j}}{(k-x)^j}$, where $\beta_{k,k}$ are non-zero for $k = 1,\ldots,5$. Indeed, $\beta_{k,k} = (k-x)^k F(x)|_{x=k} \neq 0$. It was seen in class that

$$\frac{1}{(k-x)^j} = \frac{(1/k)^j}{(1-\frac{x}{k})^j} = \sum_{n=0}^{\infty} \binom{n+j-1}{j-1} \frac{1}{k^{n+j}} x^n.$$

Putting everything together, we get

$$F(x) = \sum_{n=0}^{\infty} x^n \sum_{k=1}^{5}\sum_{j=1}^{k} \beta_{k,j} \binom{n+j-1}{j-1} \frac{1}{k^{n+j}},$$

which is the generating function of the sequence $a_n = \sum_{k=1}^{5}\sum_{j=1}^{k} \beta_{k,j} \binom{n+j-1}{j-1} \frac{1}{k^{n+j}}$. This sequence is $\Theta(1)$ because of the term corresponding to $k = j = 1$. All other terms are $O(n^{j-1}2^{-n})$. Note that the binomial coefficient $\binom{n+j-1}{j-1}$ is $O(n^{j-1})$, which grows much slower than the exponential k^{n+j} in the denominator.

b) Multiplying $F(x)$ with x corresponds to a shift of one position to the right, which does not affect the growth of the sequence, i.e., the sequence with generating function $G(x)$ is $b_n = \begin{cases} a_{n-1}, & \text{when } n > 0 \\ 0, & \text{when } n = 0 \end{cases}$. Therefore, the answer is also $\Theta(1)$.

c) We have

$$H(x) = \frac{1-x}{\prod_{k=1}^{5}(k-x)^k} = \prod_{k=2}^{5} \frac{1}{(k-x)^k}.$$

In a similar manner to point a), we obtain

$$H(x) = \sum_{n=0}^{\infty} x^n \sum_{k=2}^{5}\sum_{j=1}^{k} \gamma_{k,j} \binom{n+j-1}{j-1} \frac{1}{k^{n+j}}.$$

Note that the only difference is that now k starts at 2. The term that increases fastest is the one corresponding to $k = j = 2$, i.e., $(n+1)2^{-(n+2)}$. Using the fact that $\gamma_{2,2}$ is non-zero, we conclude that the sequence is $\Theta(n2^{-n})$.

Problem 2. The problem is equivalent to finding the coefficient of x^n in the formal series

$$F(x) = (1+x+x^2+\ldots)(1+x+x^2+\ldots)(1+x^2+x^4+\ldots) = \frac{1}{(1-x)^2}\frac{1}{1-x^2}.$$

By using partial fraction expansion, there are numbers $\alpha, \beta, \gamma, \delta$ such that

$$\begin{aligned} F(x) &= \frac{\alpha}{1-x} + \frac{\beta}{(1-x)^2} + \frac{\gamma}{(1-x)^3} + \frac{\delta}{1+x} \\ &= \frac{\alpha(1-x)^2(1+x) + \beta(1-x^2) + \gamma(1+x) + \delta(1-x)^3}{(1-x)^3(1+x)}. \end{aligned}$$

We identify the polynomial in the numerator with 1 and we obtain the system of equations

$$\begin{cases} \alpha+\beta+\gamma+\delta = 1 \\ -\alpha+\gamma-3\delta = 0 \\ -\alpha-\beta+3\delta = 0 \\ \alpha-\delta = 0 \end{cases},$$

whose resolution (by elimination, for example) gives $\alpha = \delta = 1/8$, $\beta = 1/4$ and $\gamma = 1/2$.

We retrieve

$$\begin{aligned} F(x) &= \frac{1/8}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/2}{(1-x)^3} + \frac{1/8}{1+x} \\ &= \frac{1}{8}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n x^n \\ &= \sum_{n=0}^{\infty} x^n \left(\frac{1}{8}((-1)^n+1) + \frac{1}{4}(n+1) + \frac{1}{2}\binom{n+2}{2}\right), \end{aligned}$$

from where we can read off each coefficient, so the answer is

$$a_n = \frac{1}{8}((-1)^n+1) + \frac{1}{4}(n+1) + \frac{1}{2}\binom{n+2}{2} = \left\lceil \frac{(n+3)(n+1)}{4} \right\rceil.$$

Do not forget to always check your result by computing a_n for a couple of indices by hand!

Problem 3.

i) The probability that a fair coin lands Heads in a single trial is $\frac{1}{2}$. As the $2n$ trials are independent and in each of them the probability of obtaining Heads is $\frac{1}{2}$, the probability of having $2n$ Heads is $\frac{1}{2^{2n}}$.

   Another way to see the result is the following. There are 2^{2n} possible outcomes which are all equiprobable. Only 1 consists of $2n$ Heads. Hence, the required probability is $\frac{1}{2^{2n}}$.

ii) There are 2^{2n} possible outcomes which are all equiprobable. In $\binom{2n}{2}$ of them, we obtain 2 Tails. Hence, the required probability is $\frac{\binom{2n}{2}}{2^{2n}}$.

iii) Generalizing the argument above, we have that the probability of observing k Tails is given by

$$p(k) = \frac{\binom{2n}{k}}{2^{2n}}.$$

We should bet on $\tilde{k}$ s.t. $p(k)$ attains its maximum at $\tilde{k}$. Let us consider the ratio $\dfrac{p(k+1)}{p(k)}$. After some simplifications, we obtain that

$$\frac{p(k+1)}{p(k)} = \frac{2n-k}{k+1} \geq 1 \Longleftrightarrow k \leq n - \frac{1}{2}.$$

Recalling that k and n are integers, we deduce that $p(k+1) \geq p(k)$ for $k < n$ and that $p(k+1) \leq p(k)$ for $k \geq n$. Hence, $p(k)$ attains its maximum when $\tilde{k} = n$ and you should bet on n Tails.

iv) The probability of getting Heads is $1 - \frac{1}{3} = \frac{2}{3}$. As the $2n$ trials are independent and in each of them the probability of obtaining Heads is $\frac{2}{3}$, the probability of having $2n$ Heads is $\left(\dfrac{2}{3}\right)^{2n}$. In addition, in $\dbinom{2n}{2}$ cases, we obtain 2 Tails and each of these cases occurs with probability $\left(\dfrac{2}{3}\right)^{2n-2} \cdot \left(\dfrac{1}{3}\right)^2$. Hence, the probability of getting 2 Tails out of $2n$ trials is $\dfrac{2^{2n-2}\binom{2n}{2}}{3^{2n}}$.

Problem 4. Let us denote by A the event that urn 1 is chosen and by B the event that the token is blue. Then, we are required to compute the conditional probability $\mathbb{P}(A|B)$. By Bayes theorem we obtain that

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|\overline{A})\mathbb{P}(\overline{A})},$$

where $\overline{A}$ denotes the complementary event to A, i.e., the event that urn 2 is chosen. Now, all the probabilities in the previous formula can be easily computed: $\mathbb{P}(A) = \dfrac{1}{3}$, $\mathbb{P}(\overline{A}) = \dfrac{2}{3}$, $\mathbb{P}(B|A) = \dfrac{1}{5}$, and $\mathbb{P}(B|\overline{A}) = \dfrac{4}{5}$. Hence, we conclude that $\mathbb{P}(A|B) = \dfrac{1}{9}$.

Problem 5.

a1) The probability of getting a sum equal to 2 is $p_1 q_1$. This probability is also equal to $\frac{1}{11}$, since the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$. Therefore,

$$p_1 q_1 = \frac{1}{11}. \tag{1}$$

a2) The probability of getting a sum equal to 12 is $p_6 q_6$. This probability is also equal to $\frac{1}{11}$, since the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$. Therefore,

$$p_6 q_6 = \frac{1}{11}.$$

a3)

$$\frac{a+b}{2} \geq \sqrt{ab} \Longleftrightarrow a + b \geq 2\sqrt{ab} \Longleftrightarrow (a+b)^2 \geq 4ab,$$

where the last $\Longleftrightarrow$ is allowed because a and b are non-negative and, therefore, their sum is non-negative. In addition,

$$(a+b)^2 - 4ab = (a-b)^2 \geq 0,$$

which is enough to prove the desired inequality.

a4) Let s be the probability that the sum of the outcomes of the two dice is 7. Then,

$$s = p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \geq p_1q_6 + p_6q_1 = \frac{1}{11}\left(\frac{p_1}{p_6} + \frac{p_6}{p_1}\right),$$

where the last equality comes from points a1) and a2). Using point a3), we also obtain that

$$\frac{p_1}{p_6} + \frac{p_6}{p_1} \geq 2\sqrt{\frac{p_1}{p_6} \cdot \frac{p_6}{p_1}} = 2.$$

Consequently, $s \geq \frac{2}{11}$. In addition, s must be equal to $\frac{1}{11}$, because the sum of the outcomes is uniform in $\{2, 3, \cdots, 12\}$, from which we obtain a contradiction.

b) (*Bonus*) The generating function of the sum of independent random variables is the product of the individual generating functions, i.e., $s(x) = p(x)q(x)$. In addition, using the fact that the sum of outcomes is uniform in $\{2, 3, \cdots, 12\}$, we obtain that

$$s(x) = \frac{1}{11}\sum_{i=2}^{11} x^i = \frac{1}{11} \cdot x^2 \cdot \frac{x^{11} - 1}{x - 1}.$$

The polynomial $\frac{x^{11} - 1}{x - 1}$ has no real roots: the fact that $x = 1$ is not a root can be easily checked by euclidean division; if $x > 1$, then $x^{11} - 1 > x - 1$ and the inequality is reversed for $x < 1$. On the other hand, $p(x)$ is equal to x times a polynomial of odd degree and the same reasoning applies to $q(x)$. Hence, $s(x) = p(x)q(x)$ has two roots in 0 plus two other real roots, which is a contradiction.