Discrete Structures EPFL, Fall 2013

Solution to Problem Set 11

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Problem 1. The first coin/bill deposited can be one of three things. There are a_{n-1} ways to pay if the sequence begins with a \$1 coin; there are also a_{n-1} ways to pay if the sequence begins with a \$1 bill; and there are a_{n-2} ways to pay if the sequence begins with a \$2 bill. Thus we can write the recursion

$$a_n = 2a_{n-1} + a_{n-2}, \text{ for } n \geq 2.$$

The initial conditions are $a_0 = 1$ (there is only one way to pay nothing) and $a_1 = 2$ (1 dollar can be paid either by coin or by bill).

The characteristic equation associated to the recursion is $r^2 - 2r - 1$, which has characteristic roots $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$. Hence, the solution has the form $a_n = \alpha r_1^n + \beta r_2^n$. By plugging in the initial conditions $a_0 = 1$ and $a_1 = 2$, we obtain the system

$$\begin{cases} \alpha + \beta = 1, \\ \alpha(1+\sqrt{2}) + \beta(1-\sqrt{2}) = 2. \end{cases}$$

The solution is given by $\alpha = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ and $q = \frac{1}{2} - \frac{1}{2\sqrt{2}}$, so, for any $n \in \mathbb{N}$,

$$a_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(1+\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(1-\sqrt{2})^n.$$

Problem 2.

i) The associated characteristic equation is $r^2 - 5r + 4$, which has the characteristic roots $r_1 = 1$ and $r_2 = 4$. Hence, $a_n = \alpha \cdot 1^n + \beta \cdot 4^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 1$ and $a_1 = 0$, we obtain $\alpha = 4/3$ and $\beta = -1/3$, which yields the solution for $n \in \mathbb{N}$

$$a_n = \frac{4}{3} - \frac{1}{3} \cdot 4^n.$$

ii) As in the previous case, the characteristic equation is again $r^2 - 5r + 4$, which has the characteristic roots $r_1 = 1$ and $r_2 = 4$. Hence, $a_n = \alpha \cdot 1^n + \beta \cdot 4^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 0$ and $a_1 = 1$, we obtain $\alpha = -1/3$ and $\beta = 1/3$, which yields the solution for $n \in \mathbb{N}$

$$a_n = -\frac{1}{3} + \frac{1}{3} \cdot 4^n.$$

iii) The associated characteristic equation is $r^2 + 10r + 21$, which has the characteristic roots $r_1 = -3$ and $r_2 = -7$. Hence, $a_n = \alpha \cdot (-3)^n + \beta \cdot (-7)^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 2$ and $a_1 = 1$, we obtain $\alpha = 15/4$ and $\beta = -7/4$, which yields the solution for $n \in \mathbb{N}$

$$a_n = \frac{15}{4} \cdot (-3)^n - \frac{7}{4} \cdot (-7)^n.$$

iv) The associated characteristic equation is $r^2 - 1$, which has the characteristic roots $r_1 = 1$ and $r_2 = -1$. Hence, $a_n = \alpha \cdot 1^n + \beta \cdot (-1)^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 2$ and $a_1 = -1$, we obtain $\alpha = 1/2$ and $\beta = 3/2$, which yields the solution for $n \in \mathbb{N}$

$$a_n = \frac{1}{2} + \frac{3}{2} \cdot (-1)^n.$$

v) The associated characteristic equation is $r^2 - 2r - 2$, which has the characteristic roots $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$. Hence, $a_n = \alpha \cdot (1 + \sqrt{3})^n + \beta \cdot (1 - \sqrt{3})^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 0$ and $a_1 = 1$, we obtain $\alpha = \sqrt{3}/6$ and $\beta = -\sqrt{3}/6$, which yields the solution for $n \in \mathbb{N}$

$$a_n = \frac{\sqrt{3}}{6} \cdot (1 + \sqrt{3})^n - \frac{\sqrt{3}}{6} \cdot (1 - \sqrt{3})^n.$$

vi) The associated characteristic equation is $r^2 - 2r + 1$, which has the characteristic roots $r_1 = r_2 = 1$. Hence, $a_n = \alpha \cdot 1^n + \beta \cdot n \cdot 1^n$, where α and β are to be determined according to the initial conditions. Substituting $a_0 = 3$ and $a_1 = 5$, we obtain $\alpha = 3$ and $\beta = 2$, which yields the solution for $n \in \mathbb{N}$

$$a_n = 3 + 2n.$$

vii) The associated characteristic equation is $r^3 + 6r^2 + 11r + 6$, which has the characteristic roots $r_1 = -1$, $r_2 = -2$, and $r_3 = -3$. Hence, $a_n = \alpha \cdot (-1)^n + \beta \cdot (-2)^n + \gamma \cdot (-3)^n$, where α, β, and γ are to be determined according to the initial conditions. Substituting $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$, we obtain $\alpha = 7/2$, $\beta = -6$, and $\gamma = 5/2$, which yields the solution for $n \in \mathbb{N}$

$$a_n = \frac{7}{2} \cdot (-1)^n - 6 \cdot (-2)^n + \frac{5}{2} \cdot (-3)^n.$$

viii) The associated characteristic equation is $r^4 - 10r^3 + 37r^2 - 60r + 36$, which has the characteristic roots $r_1 = r_2 = 2$, and $r_3 = r_4 = 3$. Hence, $a_n = \alpha \cdot 2^n + \beta \cdot n \cdot 2^n + \gamma \cdot 3^n + \delta \cdot n \cdot 3^n$, where α, β, γ, δ are to be determined according to the initial conditions. Substituting $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$, we obtain $\alpha = -15$, $\beta = -4$, $\gamma = 15$, and $\delta = -7/3$which yields the solution for $n \in \mathbb{N}$

$$a_n = -15 \cdot 2^n - 4 \cdot n \cdot 2^n + 15 \cdot 3^n - \frac{7}{3} \cdot n \cdot 3^n.$$

Problem 3.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Observe that, for any $k \in \mathbb{N}$, $x^k G(x) = \sum_{n=k}^{\infty} a_{n-k} x^n$.

i) From the recurrence relation, we see that

$$\begin{aligned} G(x) - 5xG(x) + 4x^2G(x) &= \sum_{n=0}^{\infty} a_n x^n - 5 \sum_{n=1}^{\infty} a_{n-1} x^n + 4 \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= a_0 + a_1 x - 5a_0 x + \sum_{n=2}^{\infty} (a_n - 5a_{n-1} + 4a_{n-2}) x^n = 1 - 5x. \end{aligned}$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{1 - 5x}{1 - 5x + 4x^2} = \frac{1 - 5x}{(1 - x)(1 - 4x)} = \frac{4/3}{1 - x} - \frac{1/3}{1 - 4x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \frac{4}{3}\sum_{n=0}^{\infty} x^n - \frac{1}{3}\sum_{n=0}^{\infty} 4^n x^n \qquad \Longrightarrow \qquad a_n = \frac{4}{3} - \frac{1}{3}\cdot 4^n.$$

ii) From the recurrence relation, we see that

$$G(x) - 5xG(x) + 4x^2G(x) = \sum_{n=0}^{\infty} a_n x^n - 5\sum_{n=1}^{\infty} a_{n-1}x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n$$
$$= a_0 + a_1 x - 5a_0 x + \sum_{n=2}^{\infty}(a_n - 5a_{n-1} + 4a_{n-2})x^n = x.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{x}{1-5x+4x^2} = \frac{x}{(1-x)(1-4x)} = -\frac{1/3}{1-x} + \frac{1/3}{1-4x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = -\frac{1}{3}\sum_{n=0}^{\infty} x^n + \frac{1}{3}\sum_{n=0}^{\infty} 4^n x^n \qquad \Longrightarrow \qquad a_n = -\frac{1}{3} + \frac{1}{3}\cdot 4^n.$$

iii) From the recurrence relation, we see that

$$G(x) + 10xG(x) + 21x^2G(x) = \sum_{n=0}^{\infty} a_n x^n + 10\sum_{n=1}^{\infty} a_{n-1}x^n + 21\sum_{n=2}^{\infty} a_{n-2}x^n$$
$$= a_0 + a_1 x + 10a_0 x + \sum_{n=2}^{\infty}(a_n + 10a_{n-1} + 21a_{n-2})x^n = 2 + 21x.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{2+21x}{1+10x+21x^2} = \frac{2+21x}{(1+3x)(1+7x)} = \frac{15/4}{1+3x} - \frac{7/4}{1+7x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \frac{15}{4}\sum_{n=0}^{\infty}(-3)^n x^n - \frac{7}{4}\sum_{n=0}^{\infty}(-7)^n x^n \qquad \Longrightarrow \qquad a_n = \frac{15}{4}\cdot(-3)^n - \frac{7}{4}\cdot(-7)^n.$$

iv) From the recurrence relation, we see that

$$G(x) - x^2G(x) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-2}x^n$$
$$= a_0 + a_1 x + \sum_{n=2}^{\infty}(a_n - a_{n-2})x^n = 2 - x.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{2-x}{1-x^2} = \frac{2-x}{(1-x)(1+x)} = \frac{1/2}{1-x} + \frac{3/2}{1+x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n x^n \qquad \Longrightarrow \qquad a_n = \frac{1}{2} + \frac{3}{2}\cdot(-1)^n.$$

v) From the recurrence relation, we see that

$$G(x) - 2xG(x) - 2x^2G(x) = \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n - 2\sum_{n=2}^{\infty} a_{n-2}x^n$$
$$= a_0 + a_1 x - 2a_0 x + \sum_{n=2}^{\infty}(a_n - 2a_{n-1} - 2a_{n-2})x^n = x.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{x}{1-2x-2x^2} = \frac{x}{(1-(1+\sqrt{3})x)(1-(1-\sqrt{3})x)} = \frac{\sqrt{3}/6}{1-(1+\sqrt{3})x} - \frac{\sqrt{3}/6}{1-(1-\sqrt{3})x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \frac{\sqrt{3}}{6}\sum_{n=0}^{\infty}(1+\sqrt{3})^n x^n - \frac{\sqrt{3}}{6}\sum_{n=0}^{\infty}(1-\sqrt{3})^n x^n \quad \Longrightarrow \quad a_n = \frac{\sqrt{3}}{6}\cdot(1+\sqrt{3})^n - \frac{\sqrt{3}}{6}\cdot(1-\sqrt{3})^n.$$

vi) From the recurrence relation, we see that

$$G(x) - 2xG(x) + x^2G(x) = \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n$$
$$= a_0 + a_1 x - 2a_0 x + \sum_{n=2}^{\infty}(a_n - 2a_{n-1} + a_{n-2})x^n = 3 - x.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{3-x}{1-2x+x^2} = \frac{3-x}{(1-x)^2} = \frac{1}{1-x} + \frac{2}{(1-x)^2}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}(n+1)x^n \qquad \Longrightarrow \qquad a_n = 3 + 2n.$$

vii) From the recurrence relation, we see that

$$G(x) + 6xG(x) + 11x^2G(x) + 6x^3G(x) = \sum_{n=0}^{\infty} a_n x^n + 6\sum_{n=1}^{\infty} a_{n-1}x^n + 11\sum_{n=2}^{\infty} a_{n-2}x^n + 6\sum_{n=3}^{\infty} a_{n-3}x^n$$
$$= a_0 + a_1 x + a_2 x^2 + 6a_0 x + 6a_1 x^2 + 11a_0 x^2 \sum_{n=3}^{\infty}(a_n + 6a_{n-1} + 11a_{n-2} + 6a_{n-3})x^n = x + 8x^2.$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{x+8x^2}{1+6x+11x^2+6x^3} = \frac{x+8x^2}{(1+x)(1+2x)(1+3x)} = \frac{7/2}{1+x} - \frac{6}{1+2x} + \frac{5/2}{1+3x}.$$

As a result, for $n \in \mathbb{N}$,

$$G(x) = \frac{7}{2}\sum_{n=0}^{\infty}(-1)^n x^n - 6\sum_{n=0}^{\infty}(-2)^n x^n + \frac{5}{2}\sum_{n=0}^{\infty}(-3)^n x^n \quad \Longrightarrow \quad a_n = \frac{7}{2}\cdot(-1)^n - 6\cdot(-2)^n + \frac{5}{2}\cdot(-3)^n.$$

viii) From the recurrence relation, we see that

$$
\begin{aligned}
&G(x) - 10xG(x) + 37x^2G(x) - 60x^3G(x) + 36x^4G(x)\\
=&\sum_{n=0}^{\infty} a_nx^n - 10\sum_{n=1}^{\infty} a_{n-1}x^n + 37\sum_{n=2}^{\infty} a_{n-2}x^n - 60\sum_{n=3}^{\infty} a_{n-3}x^n + 36\sum_{n=4}^{\infty} a_{n-4}x^n\\
=&a_0 + a_1x + a_2x^2 + a_3x^3 - 10a_0x - 10a_1x^2 - 10a_2x^3 + 37a_0x^2 + 37a_1x^3\\
&- 60a_0x^3 \sum_{n=3}^{\infty}(a_n - 10a_{n-1} + 37a_{n-2} - 60a_{n-3} + 36a_{n-4})x^n = x^2 - 10x^3.
\end{aligned}
$$

Therefore, expanding $G(x)$ into partial fractions, we obtain that

$$G(x) = \frac{x^2 - 10x^3}{1 - 10x + 37x^2 - 60x^3 + 36x^4} = \frac{x^2 - 10x^3}{(1-2x)^2(1-3x)^2} = -\frac{11}{1-2x} - \frac{4}{(1-2x)^2} + \frac{52/3}{1-3x} - \frac{7/3}{(1-3x)^2}.$$

As a result,

$$G(x) = -11\sum_{n=0}^{\infty} 2^nx^n - 4\sum_{n=0}^{\infty}(n+1)2^nx^n + \frac{52}{3}\sum_{n=0}^{\infty} 3^nx^n - \frac{7}{3}\sum_{n=0}^{\infty}(n+1)3^nx^n,$$

which implies that, for $n \in \mathbb{N}$,

$$a_n = -15 \cdot 2^n - 4 \cdot n \cdot 2^n + 15 \cdot 3^n - \frac{7}{3} \cdot n \cdot 3^n.$$

Problem 4.

(a) $\dfrac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots$. The coefficient of x^8 is 2^8.

(b) $\dfrac{x^3}{1-3x} = x^3(1 + 3x + (3x)^2 + (3x)^3 + \ldots)$. The coefficient of x^8 is 3^5.

(c) $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$. The coefficient of x^8 is 9. There are more ways to derive this. One is

$$\left(\frac{1}{(1-x)}\right)^2 = \left(\sum_{j=0}^{\infty} x^j\right)^2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^jx^k = \sum_{s=0}^{\infty}\sum_{k=0}^{s} x^{s-k}x^k = \sum_{s=0}^{\infty} x^s \sum_{k=0}^{s} 1 = \sum_{s=0}^{\infty}(s+1)x^s.$$

Another way is to observe that $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$. We first write

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{s=0}^{\infty} x^s.$$

Taking derivative on both sides, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{s=0}^{\infty}(s+1)x^s.$$

Note that the infinite sum contains a limit and we shamelessly swapped the order of derivation (which is another kind of limit) and the limit in the infinite sum. This could be dangerous in general, as swapping limits not always works, but in certain cases it can be made rigorous. If you just need an answer that you can later prove by other means, like induction, this method can be used to do the guesswork.

(d) By the substitution $x \to -2x$ in the previous formulas we get

$$\frac{1}{(1+2x)^2} = 1 + 2 \cdot (-2x) + 3 \cdot (-2x)^2 + \ldots = \sum_{k=0}^{\infty} (k+1)(-2)^k x^k.$$

Then

$$\frac{x^2}{(1+2x)^2} = \sum_{k=0}^{\infty} (k+1)(-2)^k x^{k+2}.$$

We read off the coefficient for $k + 2 = 8$, which is $7(-2)^6$.

(e) This can be done directly, by putting $y = 3x^2$ and expanding $1/(1-y)$:

$$\frac{1}{1-3x^2} = 1 + 3x^2 + 9x^4 + \ldots = \sum_{k=0}^{\infty} 3^k x^{2k},$$

We get the same answer if we do the partial fraction decomposition:

$$\begin{aligned}\frac{1}{1-3x^2} &= \frac{1/2}{1+x\sqrt{3}} + \frac{1/2}{1-x\sqrt{3}} \\ &= \frac{1}{2}(1 - x\sqrt{3} + 3x^2 - 3\sqrt{3}x^3 + \ldots) + \frac{1}{2}(1 + x\sqrt{3} + 3x^2 + 3\sqrt{3}x^3 + \ldots) \\ &= 1 + 3x^2 + 9x^4 + \ldots.\end{aligned}$$

The coefficient of x^8 is 3^4.

(f) By partial fraction decomposition, the formula can be written as

$$x^3 \left(\frac{1}{1-2x} - \frac{3}{1+x} + \frac{7}{1-x} \right).$$

Since $\frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k$, $\frac{3}{1+x} = 3\sum_{k=0}^{\infty} (-1)^k x^k$, and $\frac{7}{1-x} = 7\sum_{k=0}^{\infty} x^k$, we add the contribution of each for $k = 5$ and we get 42, which is the answer.