Discrete Structures EPFL, Fall 2013

Solution to Graded Problem Set 10

Date: 22.11.2013 Due date: 29.11.2013, noon

Problem 1.

(a) By the product rule, there are $5 \times 3 \times 10 \times 5 \times 4 = 3000$ different special dinner.

(b) There are 6! ways to rearrange the word GOOGOL but we need to eliminate the repetitions (when an O is switched with another O for example). There are 3 O's and 2 G's, so we get $\frac{6!}{3!2!}$ possible distinct rearrangements.

(c) i) We choose 6 people among $20 + 17 = 37$ without replacement and the selection is unordered. This can be done in $\binom{37}{6}$ ways.

ii) We select 3 women among 20 and 3 men among 17. The selections are unordered and without replacement. By the product rule, this can be done in $\binom{20}{3} \cdot \binom{17}{3}$ ways.

iii) There are two ways to find the solution. The first is to find the number of committees for exactly 2, 3, 4, 5 or 6 and sum them all. The second is to subtract committees with no men or exactly one man from the number of all possible committees. Therefore, the answer is

$$\sum_{i=2}^{6} \binom{17}{i}\binom{20}{6-i} = \binom{37}{6} - \binom{20}{6} - 17\binom{20}{5}.$$

iv) There are $\binom{17}{6}$ all-men committees and $\binom{20}{6}$ all-women committees, so the answer is $\binom{17}{6} + \binom{20}{6}$ by the sum rule.

(d) There are 2^k ways to choose the values of the variables and on each of these choices the proposition can take the values T or F. Thus, by the product rule, there are 2^{2^k} propositions.

Problem 2. The same hand is counted twice. For example, getting the kings of hearts and diamonds first and the sevens of clubs and hearts second is the same as getting the pair of sevens first and the pair of kings second. To obtain the correct answer, divide the given answer by 2.

Problem 3.

(a) The number of codes is $26 \cdot 10^3 = 26,000$. Since $\lceil 60,000/26,000 \rceil = 3$, by pigeonhole at least three parts have the same code number.

(b) Divide the square into four 1×1 squares. At least two of the five points lie in or on the edge of one of these 1×1 squares. The maximum distance between two such points is $\sqrt{2}$.

Problem 4. We propose two distinct proofs of the claim.

First proof: binomial theorem. The number of binary strings with an even number of 1's is $\sum_{\substack{k=1 \\ k \text{ even}}}^{n} \binom{n}{k}$ by the sum rule. Observe that $\frac{1+(-1)^k}{2}$ is equal to 1 if k is even and is equal to 0 otherwise. Remark also that by the binomial theorem we have that $\sum_{k=1}^{n} \binom{n}{k} = (1+1)^n = 2^n$ and that $\sum_{k=1}^{n} (-1)^k \binom{n}{k} = \sum_{k=1}^{n} (-1)^k 1^{n-k} \binom{n}{k} = (1-1)^n = 0$. Putting all these steps together, we obtain

$$\sum_{\substack{k=1 \\ k \text{ even}}}^{n} \binom{n}{k} = \sum_{k=1}^{n} \frac{1+(-1)^k}{2} \binom{n}{k} = \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} + \frac{1}{2} \sum_{k=1}^{n} (-1)^k \binom{n}{k} = 2^{n-1} + \frac{(1-1)^n}{2} = 2^{n-1},$$

which is exactly half of the total number of binary strings of length n. The other half of the binary strings of length n contains an odd number of 1's and the proof is complete.

Second proof: bijection. Let A be the set of binary strings of length n with an even number of 1's and let B be the set of binary strings of length n with an odd number of 1's. Define $f : A \to B$ as the function which flips the first bit of the binary string given as input. Formally, let $w = (w_1, w_2, \ldots, w_n) \in A$, then $f(w) = w'$, where $w' = (1 - w_1, w_2, \ldots, w_n)$. It can be verified that f is a function and, moreover, a bijection. Then, A and B have the same cardinality and the claim is proved.

Problem 5.

(a) The number of increasing functions from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$ equals the combinations of k elements chosen from a class of n without replacement. Indeed, we need to count the choices for the images of the k elements of the domain into the n elements of the codomain. The selection is unordered, because the order is fixed by the monotonicity of the function. In addition, no repetitions are allowed, since the functions are increasing, and, therefore, injective. As a result,

$$|\mathcal{F}| = \binom{n}{k}.$$

(b) The number of non-decreasing functions from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$ equals the combinations of k elements chosen from a class of n with replacement. Indeed, in this case, we need to count the choices for the images of the k elements of the domain into the n elements of the codomain, where the selection is still unordered because the order is fixed by the monotonicity of the function, but repetitions are allowed, because the functions are non-decreasing, i.e., two distinct elements of the domain can be mapped into the same element of the codomain. As a result,

$$|\mathcal{G}| = \binom{n+k-1}{k}.$$

Another way to prove this result consists in associating to each non-decreasing function g the function h, defined as

$$h(i) = g(i) + i - 1.$$

It is easy to check that h goes from $\{1, \cdots, k\}$ to $\{1, \cdots, n+k-1\}$ and is increasing. Basically, we are defining a bijection between $\mathcal{G}$ and the set $\mathcal{H}$ of increasing functions $h : \{1, \ldots, k\} \to \{1, \ldots, n+k-1\}$. Then, $|\mathcal{G}| = |\mathcal{H}| = \binom{n+k-1}{k}$, where the last equality comes from point (a).

Problem 6. First of all, let us compute in two ways the product of all the elements of a matrix which satisfies the required conditions (product of the elements of each row equal to -1 and product of the elements of each column equal to -1). Since there are m rows, then the product of the elements of the matrix done row by row is $(-1)^m$. In addition, there are n columns, so the product of the elements of the matrix done column by column is $(-1)^n$. Consequently, if m and n do not have the same parity, i.e. $n \not\equiv m \pmod 2$, then there is no matrix with the required property.

Let $n \equiv m \pmod 2$. Consider the submatrix A_s obtained removing the last row and the last column of the original matrix A. Clearly, there are $2^{(n-1)(m-1)}$ possible ways of choosing this submatrix. Denote by r_i, $i \in \{1, \dots, m-1\}$, the product of the elements of the i-th row of the submatrix. Now, the last element of each row of A has to be set to $-r_i$, in order to ensure that the product of the elements of each row is -1. Analogously, denote by c_i, $i \in \{1, \dots, n-1\}$, the product of the elements of the i-th column of the submatrix. Now, the last element of each column of A has to be set to $-c_i$, in order to ensure that the product of the elements of each column is -1. With this procedure, only one element is left, namely the element in the lower right corner x_{nm} (the element at the intersection between the m-th row and the n-th column). This value has to be chosen so that the product of the elements of the last row is -1 and the product of the elements of the last column is -1. In formulas,

$$x_{nm} \cdot \prod_{i=1}^{m-1} (-r_i) = -1, \tag{1}$$

$$x_{nm} \cdot \prod_{i=1}^{n-1} (-c_i) = -1. \tag{2}$$

Therefore, since the elements of the matrix are in $\{-1, +1\}$,

$$x_{nm} = -\prod_{i=1}^{m-1} (-r_i) = (-1)^m \prod_{i=1}^{m-1} r_i, \tag{3}$$

$$x_{nm} = -\prod_{i=1}^{n-1} (-c_i) = (-1)^n \prod_{i=1}^{n-1} c_i. \tag{4}$$

Observe that $\prod_{i=1}^{m-1} r_i = \prod_{i=0}^{n-1} c_i$, since both the expressions are equal to the product of all the elements of the submatrix A_s (done row by row and column by column, respectively). In addition, as $n \equiv m \pmod 2$, then $(-1)^m = (-1)^n$ and the two conditions (3) and (4) reduce to the same equation. This equation fixes the value of x_{nm}.

As a result, if $n \not\equiv m \pmod 2$, there is no matrix which fulfills the required condition. Otherwise, there are $2^{(m-1)(n-1)}$ possible matrices.