Discrete Structures EPFL, Fall 2013

Solution to Special Problem Set 8

Date: 8.11.2013 Not graded

Problem.

(a) **Base step.** Let $n = 1$. Since $\sum_{i=1}^{1} f_i^2 = f_i^2 = 1$ and $f_1 f_2 = 1$, then the assertion is true.

Induction step. Assume that $\sum_{i=1}^{n} f_i^2 = f_n \cdot f_{n+1}$ for some $n \geq 1$. Then, using the induction hypothesis and the fact that $f_{n+2} = f_{n+1} + f_n$, we obtain that

$$\sum_{i=1}^{n+1} f_i^2 = \sum_{i=1}^{n} f_i^2 + f_{n+1}^2 = f_n \cdot f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2}$$

(b) **Base step.** Let $n = 1$. Since $f_2 f_0 - f_1^2 = 0 - 1 = -1$ and $(-1)^1 = -1$, then the assertion is true.

Induction step. Assume that $f_{n+1} \cdot f_{n-1} - f_n^2 = (-1)^n$ for some $n \geq 1$. Using the recursive definition $f_{n+2} = f_{n+1} + f_n$, $f_{n+1} = f_n + f_{n-1}$ and applying the induction hypothesis, we obtain

$$\begin{aligned} f_{n+2} f_n - f_{n+1}^2 &= (f_{n+1} + f_n) f_n - f_{n+1}(f_n + f_{n-1}) \\ &= f_{n+1} f_n + f_n^2 - f_{n+1} f_n - f_{n+1} f_{n-1} \\ &= -(f_{n+1} f_{n-1} - f_n^2) \\ &= (-1) \cdot (-1)^n \\ &= (-1)^{n+1}. \end{aligned}$$

(c) **Base step.** Let $n = 1$. Observing that $\begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and recalling the definition of A, we easily check the equality.

Induction step. Assume that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$. Then, using the recursive definition $f_{n+2} = f_{n+1} + f_n, f_{n+1} = f_n + f_{n-1}$ and applying the induction hypothesis, we obtain

$$A^{n+1} = A^n A = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{n+1} + f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}.$$

Solution to mock midterm. The numbers indicate the position of the correct answer to each of the questions.

1. (a) 3 – (b) 2 – (c) 2
2. (a) 4 – (b) 2 – (c) 3
3. (a) 3 – (b) 2 – (c) 3
4. (a) 2 – (b) 4 – (c) 4
5. (a) 2 – (b) 2 – (c) 1
6. (a) 2 – (b) 3 – (c) 3