Solution to Problem Set 7

Date: 1.11.2013 Not graded

Problem 1.

a) We look at the 104 factors that appear in $104! = 1 \cdot 2 \cdot \ldots \cdot 104$. Out of those, exactly $\left\lfloor \frac{104}{5} \right\rfloor = 20$ are divisible by 5, of which $\left\lfloor \frac{104}{25} \right\rfloor = 4$ are divisible by 25. So $20 - 4 = 16$ factors contribute with 5 and 4 contribute with 5^2. Thus $5^{24} \mid 104!$ and $5^{25} \nmid 104!$.

In a similar manner, we have that $\left\lfloor \frac{104}{2} \right\rfloor = 52 > 24$ factors are divisible by two, thus $2^{24} \mid 104!$.

Since 5^{24} and 2^{24} are coprime, their product 10^{24} also divides $104!$ and furthermore $10^{25} \nmid 104!$, because $5^{25} \nmid 104!$. This means there are exactly 24 zeroes in the decimal representation of $104!$.

b) Using Fermat's Little Theorem (see last problem set), we have $25^{53-1} \equiv 1 \pmod{53}$. Since $3702298 \equiv 2 \pmod{52}$, there is $m \in \mathbb{N}$ such that $3702298 = 52m + 2$. We can then write

$$25^{3702298} \equiv 25^{52m+2} \equiv 1 \cdot 25^2 \equiv 42 \pmod{53}.$$

So the answer is 42.

c) False: $21 \equiv 1 \pmod 4$ but $7 \equiv 3 \pmod 4$ and $7 \mid 21$.

d) True. Since $p, q > 2$, there are both odd, so $p \equiv q \equiv 1 \pmod 2$. Then $p \cdot q + 1 \equiv 1 \cdot 1 + 1 \equiv 0 \pmod 2$. and so $2 \mid (p \cdot q + 1)$. Since $p \cdot q + 1 > 2$, it is not prime.

e) Let p be one of the primes. Since $p > 2$ we have $p \equiv 1 \pmod 2$. The other prime q is either $p + 1$ or $p - 1$. However, we have $q \equiv 1 + 1 \equiv 1 - 1 \equiv 0 \pmod 2$. Then $2 \mid q$ and $q > 2$ so q is not prime. Contradiction!

Problem 2. The representation $n = 2^k + j$ with $k \in \mathbb{N}$ and $j \in \{0, \ldots, 2^k - 1\}$ is unique because the binary representation of any natural number is unique. Alternatively, the uniqueness follows from the uniqueness of quotient and remainder of the division of n by 2^k. Hence,

$$\sum_{n \geq 1} \frac{1}{n} = \sum_{k \geq 1} \sum_{j=0}^{2^k - 1} \frac{1}{2^k + j} \geq \sum_{k \geq 1} \sum_{j=0}^{2^k - 1} \frac{1}{2 \cdot 2^k}$$

where the last inequality comes from the fact that $j < 2^k$. Now,

$$\sum_{k \geq 1} \sum_{j=0}^{2^k - 1} \frac{1}{2 \cdot 2^k} = \frac{1}{2} \sum_{k \geq 1} 1 = +\infty$$

As a result,

$$\sum_{n \geq 1} \frac{1}{n} = +\infty.$$

Problem 3.

a) The cardinality of $\mathcal{A}_n$ and the number of terms on the left hand side of (1) is $(n+1)^n$. By the uniqueness of the factorization, for each element m of $\mathcal{A}_n$, the term $1/m$ appears in the expansion of the product on the left. Thus, the expansion of this product is a rearrangement of the finite sum on the right.

b) Recall that for any $a \neq 1$,

$$\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}.$$

Hence,

$$1 + \frac{1}{p_j} + \ldots + \frac{1}{p_j^n} = \sum_{i=0}^{n} \frac{1}{p_j^i} = \frac{1 - \frac{1}{p_j^{n+1}}}{1 - \frac{1}{p_j}} \leq \frac{1}{1 - \frac{1}{p_j}} = \frac{p_j}{p_j - 1} = 1 + \frac{1}{p_j - 1}$$

c) Using (1), we obtain that $\ln \sum_{m \in \mathcal{A}_n} \frac{1}{m} = \ln \prod_{i=1}^{n} \sum_{j=0}^{n} \frac{1}{p_i^j}$. Since $\ln(\cdot)$ is a monotonous function, using point b), we have that

$$\ln \prod_{i=1}^{n} \sum_{j=0}^{n} \frac{1}{p_j^n} \leq \ln \prod_{i=1}^{n} \left(1 + \frac{1}{p_i - 1}\right) = \sum_{i=1}^{n} \ln \left(1 + \frac{1}{p_i - 1}\right).$$

Let us define $f(x) = \ln(x+1)$ and $g(x) = x$. Then $g(0) = f(0) = 0$ and $g'(x) = 1 > \frac{1}{1+x} = f'(x)$ for any $x \geq 0$. Consequently $f(x) \leq g(x)$ for any $x \geq 0$. As a result,

$$\sum_{i=1}^{n} \ln \left(1 + \frac{1}{p_i - 1}\right) \leq \sum_{i=1}^{n} \frac{1}{p_i - 1}$$

Putting all these results together, we obtain $\sum_{i=1}^{n} \frac{1}{p_i - 1} \geq \ln \sum_{m \in \mathcal{A}_n} \frac{1}{m}$.

d) $\{1, \ldots, n\} \subseteq \mathcal{A}_n$ because for all $j \in \{1, \ldots, n\}$, the unique factorization of j contains only primes from $\{p_1, \ldots, p_n\}$ as $p_n \geq n$. Also the multiplicity of each prime needs to be at most n, since $p_i^{n+1} \geq 2^{n+1} > n$.

This proves that

$$\ln \sum_{m \in \mathcal{A}_n} \frac{1}{m} \geq \ln \sum_{m=1}^{n} \frac{1}{m}.$$

As concerns the left hand side, note that for $j \geq 2$,

$$\frac{1}{p_j - 1} \leq \frac{1}{p_{j-1}}.$$

In addition, if $j = 1$, then

$$\frac{1}{p_1 - 1} = \frac{1}{2 - 1} = 1.$$

Hence $\sum_{j=1}^{n} \frac{1}{p_j - 1} \leq 1 + \sum_{j=2}^{n} \frac{1}{p_{j-1}} = 1 + \sum_{j=1}^{n-1} \frac{1}{p_j}$.

e) We already know that $\sum_{m=1}^{n} \frac{1}{m}$ is unbounded as $n \to +\infty$ and its logarithm too, so the left hand side of (2) diverges. Hence, $\lim_{n\to+\infty} \sum_{j=1}^{n-1} \frac{1}{p_j} = +\infty$.

Problem 4.

a) *Base case*: for $n = 2$, the equality trivially reduces to $A_1 \cap A_2 = A_1 \cap A_2$.
Induction step: Assume $A_1 \cap (A_2 \cup \ldots \cup A_n) = (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \ \ldots \ \cup (A_1 \cap A_n)$, for fixed $n \geq 2$. Thus, by the associative property of the union, and the distributive property of union and intersection,

$$\begin{aligned} A_1 \cap (A_2 \cup \ldots \cup A_{n+1}) &= A_1 \cap ((A_2 \cup \ldots \cup A_n) \cup A_{n+1}) \\ &= (A_1 \cap (A_2 \cup \ldots \cup A_n)) \cup (A_1 \cap A_{n+1}) \\ &= (A_1 \cap A_2) \cup (A_1 \cap A_2) \cup \ldots \cup (A_1 \cap A_n) \cup (A_1 \cap A_{n+1}), \end{aligned}$$

where in the last equality we have used the inductive hypothesis. In this way we are able to prove the claim for the $n + 1$ sets $A_1, \cdots A_{n+1}$.

b) *Base case*: for $n = 1$, it is true that $13 \mid 17 - 4$.
Induction step: Given $n \geq 1$, assume that $13 \mid 17^n - 4^n$. Then, there exists k such that $17^n - 4^n = 13k$. Consequently,

$$\begin{aligned} 17^{n+1} - 4^{n+1} &= 17 \cdot 17^n - 4^{n+1} = 13 \cdot 17^n + 4 \cdot 17^n - 4^{n+1} = 13 \cdot 17^n + 4 \cdot (17^n - 4^n) \\ &= 13 \cdot 17^n + 4 \cdot 13 \cdot k = 13 \cdot (17^n + 4 \cdot k). \end{aligned}$$

Hence, $13 \mid 17^{n+1} - 4^{n+1}$.

c) *Base cases*: $32 = 3 \cdot 9 + 5$, $33 = 2 \cdot 9 + 3 \cdot 5$, $34 = 9 + 5 \cdot 5$, $35 = 7 \cdot 5$, $36 = 9 \cdot 4$.
Induction step: Assume we already know how to pay k francs, $32 \leq k \leq n$. If we want to pay $n + 1$ francs, for $n \geq 36$, first pay $n - 4$ francs (we know how to do this by induction hypothesis), and add a extra coin of 5 francs.

d) *Base case*: $\sum_{i=0}^{0} (-2)^i = 1 = \frac{(-1)^0 \cdot 2 + 1}{3}$.
Induction step: Assume $\sum_{i=0}^{n} (-2)^i = \frac{(-1)^n \cdot 2^{n+1} + 1}{3}$, then:

$$\begin{aligned} \sum_{i=0}^{n+1} (-2)^i &= \sum_{i=0}^{n} (-2)^i + (-2)^{n+1} = \frac{(-1)^n \cdot 2^{n+1} + 1}{3} + (-2)^{n+1} \\ &= \frac{(-1)^n \cdot 2^{n+1} + 1 + 3 \cdot (-2)^{n+1}}{3} = \frac{(-1)^n \cdot 2^{n+1} + 3 \cdot (-1) \cdot (-1)^n \cdot 2^{n+1} + 1}{3} \\ &= \frac{(-1)^n \cdot 2^{n+1}(1-3) + 1}{3} = \frac{(-1)^n \cdot 2^{n+1}(-2) + 1}{3} \\ &= \frac{(-1)^{n+1} \cdot 2^{n+2} + 1}{3}. \end{aligned}$$

Problem 5. The error lies in the fact that we proved a single base step. Indeed, the induction step works for $n \geq 2$. In other words, we are able to infer that all sets of $n + 1$ girls contain elements of the same hair color using the fact that the same statement holds with n, **only** for

$n \geq 2$. We are not able to prove the case $n = 2$ from the case $n = 1$ simply because if we remove g and then another random girl, we are left with the empty set! Consequently, we would need to prove a second base step for $n = 2$, i.e., we should show that all pair of girls have the same hair color. This last statement is equivalent to the statement of the "theorem" and is, obviously, false.