Discrete Structures EPFL, Fall 2013

Solution to Graded Problem Set 4

Date: 11.10.2013 Due date: 18.10.2013

Problem 1.

(a) Let a_n be the salary in year n. The following relation holds:

$$a_{n+1} = a_n\Big(1 + \frac{3}{100}\Big),$$

with initial condition $a_0 = 40000$.

Hence, $a_n = a_0\Big(1 + \frac{3}{100}\Big)^n = 40000 \cdot (1.03)^n$.

(b) In a similar manner, $a_n = a_0 \cdot \left(1 + \frac{5}{100}\right)^n = 40000 \cdot (1.05)^n$.

(c) If there was no 1000 CHF raise, the final amount would be $a_0 \cdot \left(1 + \frac{2}{100}\right)^n$ exactly as in the previous cases. The extra 1000 CHF obtained in the first year gives $1000 \cdot \left(1 + \frac{2}{100}\right)^{n-1}$ at the n-th year, since it will be raised $n-1$ times by 2%. Likewise, the extra 1000 CHF obtained in the k-th year will become $1000 \cdot \left(1 + \frac{2}{100}\right)^{n-k}$ at the end.

Adding all these contributions we obtain:

$$\begin{aligned}
a_n &= a_0\Big(1 + \frac{2}{100}\Big)^n + \sum_{k=1}^{n} 1000 \cdot \left(1 + \frac{2}{100}\right)^{n-k} \\
&= a_0 \left(1 + \frac{2}{100}\right)^n + 1000 \cdot \sum_{j=0}^{n-1} \left(1 + \frac{2}{100}\right)^j \\
&= a_0 \left(1 + \frac{2}{100}\right)^n + 1000 \cdot \frac{\left(1 + \frac{2}{100}\right)^n - 1}{1 + \frac{2}{100} - 1} \\
&= 40000 \cdot (1,02)^n + 1000 \cdot \frac{(1.02)^n - 1}{\frac{2}{100}}
\end{aligned}$$

Problem 2. After n years the population of the island, according to the predictions of the model is given by

$$10000 + \sum_{i=98}^{98+n-1} i = 10000 + \sum_{i=1}^{97+n} i - \sum_{i=1}^{97} i$$

$$= 10000 + \frac{(98+n)(97+n)}{2} - \frac{97 \cdot 98}{2} = \frac{n^2}{2} + \frac{195}{2}n + 10000.$$

Hence, the population exceeds 20000 units after n years, if and only if the following condition is satisfied

$$10000 + \frac{n^2}{2} + \frac{195}{2}n > 20000 \Longleftrightarrow n^2 + 195n - 20000 > 0.$$

The associated equation is $n^2 + 195n - 20000 = 0$, which has the solutions :

$$n_{1,2} = \frac{-195 \pm \sqrt{195^2 + 4 \times 20000}}{2}.$$

Then, we need

$$n > \frac{\sqrt{195^2 + 4 \times 20000} - 195}{2} \simeq 74.27.$$

In conclusion, in $n = 75$ years, the population is going to exceed the value of 20000 units.

Problem 3. Consider the function $f : \mathbb{N} \to A$ defined as

$f(0) = e$
$f(1) = e^e$
$f(2) = e^{e^e}$
$f(3) = e^{e^{e^e}}$
$f(n) = n - 4, \qquad$ for $n \geq 4$.

It is easy to check that f is bijective. Hence $|A| = |\mathbb{N}|$.

Problem 4.

(a) Consider $f : \mathbb{R} \to (-1, 1)$, defined as

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

As showed in Problem 6 of the previous exercise session, f is bijective.

(b) Define $A = \{1/n : n \in \mathbb{Z}, n \geq 1\}$ and $B = (0, 1) \setminus A$. Note that $(0, 1) = A \cup B \setminus \{1\}$ and $(0, 1] = A \cup B$. Consider the function $f : (0, 1) \to (0, 1]$, defined as

$$\begin{cases} f(x) = x, & \text{if } x \in B \\ f(\frac{1}{n}) = \frac{1}{n-1}, & \text{for } n \in \mathbb{Z}, n \geq 2 \end{cases}.$$

Then, f is bijective, which implies the thesis.

Problem 5. Observe that $|\mathcal{P}(\emptyset)| = 1$, so we need to show that there exists a unique strictly increasing function $f : \Delta \to \Delta$ with $\Delta = \{\delta_1, ..., \delta_{2013}\}$. Without loss of generality we can assume that $\delta_1 \leq \delta_2 \leq ... \leq \delta_{2013}$.

Clearly, $f(x) = x$ satisfies the constraints, so we need to show that there exists no other strictly increasing function from Δ to Δ. Suppose by contradiction that there exists a function g such that $g(\delta_i) = \delta_j$ with $j \neq i$.

Case I: $j > i$.

There are $2013 - i$ elements of the domain that have to be sent to $2013 - j$ elements of codomain, since $\delta_k > \delta_i \Rightarrow g(\delta_k) > g(\delta_i)$. As $2013 - i > 2013 - j$, by pigeonhole, two of those elements will be mapped in the same element of the codomain. Hence, g cannot be injective and, therefore, strictly increasing. We reach a contradiction.

Case II: $j < i$.

There are $i - 1$ elements of the domain that have to be mapped in $j - 1$ elements of the codomain, because g is strictly increasing. As $i - 1 > j - 1$, by pigeonhole, g cannot be injective and, therefore, we reach a contradiction.

As a result, $f(x) = x$ is the only strictly increasing function from Δ to Δ and $|C| = 1$.

Problem 6. Clearly, $|\mathcal{F}| \geq |\mathcal{G}|$, since all the functions from $\mathbb{N}$ to $\{0, 1\}$ can be seen as functions from $\mathbb{N}$ to $\mathbb{N}$. Hence, in order to show that equality holds, it is enough to prove that $|\mathcal{F}| \leq |\mathcal{G}|$. To show that $|\mathcal{F}| \leq |\mathcal{G}|$, first observe that the elements of $\mathcal{F}$ can be seen as infinite sequences of natural numbers (or, more formally, can be put in bijection with the infinite sequences of natural numbers). Analogously, the elements of $\mathcal{G}$ can be seen as infinite binary strings. Consider a function $f : \mathcal{F} \to \mathcal{G}$ as follows. For every sequence of natural numbers $\{a_n\}_{n=0}^{\infty}$ define $f(\{a_n\})$ as the binary string

$$\underbrace{111\cdots1}_{a_0 \text{ times}}0\underbrace{111\cdots1}_{a_1 \text{ times}}0\underbrace{111\cdots1}_{a_2 \text{ times}}0\cdots$$

Then, the function f is the desired injection from $\mathcal{F}$ to $\mathcal{G}$.