Discrete Structures — EPFL, Fall 2013

Solution to Problem Set 3

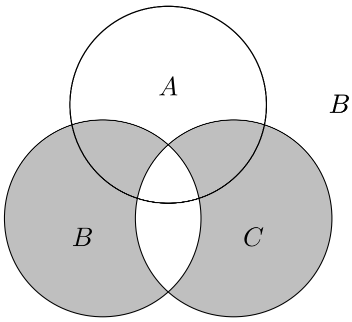
Date: 4.10.2013 — Not graded

Problem 1.

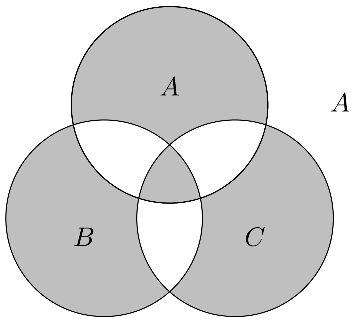
a) True:

$$\begin{aligned}
A \setminus (B \cap C) &= \{x : x \in A \wedge \neg(x \in B \wedge x \in C)\} \\
&= \{x : x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))\} \\
&= \{x : (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)\} \\
&= \{x : x \in A \setminus B \vee x \in A \setminus C\} \\
&= \{x : x \in (A \setminus B) \vee (A \setminus C)\} \\
&= A \setminus B \cup (A \setminus C)
\end{aligned}$$

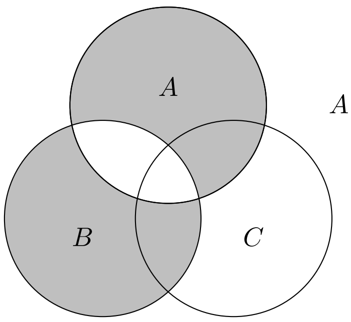
b) True:



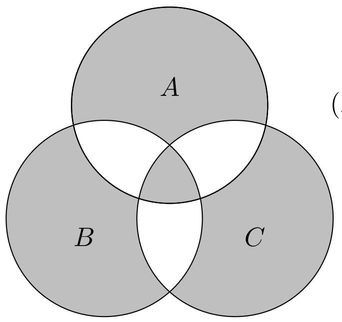
$B \oplus C$



$A \oplus (B \oplus C)$



$A \oplus B$



$(A \oplus B) \oplus C$

Problem 2.

a) $n \cdot m$

b) 2^m

c) $2^{2^n \cdot m}$

d) 2^{n+m}

e) $2\,n$

f) $n \cdot m$

Problem 3.

a) Yes, $\{\varnothing, a, \{a\}, \{\{a\}\}\}$.

b) Yes, $\{a\}$.

c) No, as the cardinality is not a power of 2.

d) Yes, $\{a, \varnothing\}$.

e) No, it lacks $\{a\}$ and $\{\varnothing\}$.

Problem 4. It can be solved using the inclusion-exclusion principle. Assume that the set A contains all the mathematicians, the set B the poets and the set C the painters. We are given that:

- $|A \cup B \cup C| = 35$
- $|A| = 18$
- $|B| = 13$
- $|C| = 14$
- $|B \cap C| = 7$
- $|A \cap C| = |A \cap B \cap C|$

Since $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$, we can conclude that $|A \cap B| = 3$.

Problem 5.

a) False, $\sqrt{2} \notin \mathbb{N}$.

b) False, $\sqrt{-1} \notin \mathbb{R}$.

c) False, $g(n)$ is not uniquely assigned.

d) False, undefined for $x = 5$.

e) True, as $\sqrt{5} \notin \mathbb{Z}$.

f) False, $F(1) = -\dfrac{1}{4} \notin \mathbb{Z}$.

g) False, not uniquely defined for $0 \leq x \leq 4$.

h) False, not defined over the whole domain.

i) False, $G(1/2) \neq G(2/4)$.

Problem 6.

a) Injective. Indeed, the function is increasing (and, therefore, injective). In addition, 3 is in the codomain, but has no preimage, which means that the function is not surjective.

b) Bijective. Indeed, the function is increasing (and, therefore, injective). In addition, the preimage of every even number is its half, and, therefore, the function is surjective.

c) Bijective. Indeed, the function is increasing (and, therefore, injective). In addition, the function is continuous and $\lim_{x\to-\infty} f(x) = -1$ and $\lim_{x\to+\infty} f(x) = +1$, which implies the surjectivity.

d) Surjective. Indeed, the function is not injective, since $f(0) = f(2\pi)$. In addition, for every $z \in \mathbb{C}$, there exist $r \in [0, \infty)$, and $\theta \in [0, 2\pi)$ s.t. $z = re^{i\theta}$. If $|z| = 1$, then $r = 1$ and $z = f(\theta)$.

e) None. Indeed, the function is not injective, since $f(2) = f(3)$. Also, the function is not surjective, since $1/2$ has no preimage.

Problem 7.

a) $\{(1,10),(2,8),(3,10),(4,8)\}$ $f \circ g : A \to C$

b) $\{(8,a),(10,b),(2,c)\}$ $f^{-1} : C \to B$

c) $\{(2,2),(8,8),(10,10)\}$ $f \circ f^{-1} : C \to C$

d) g is not injective. Indeed, $g^{-1}(a)$ is not well defined.