SOLUTIONS OF QUATUM INFORMATION AND COMPUTATIONS PROBLEMS. DOCTORAL CLASS 2013.

Homeworks September 19 - 2013

Problem 1. Mach-Zehnder Interferometer

a) We know $H|T\rangle = \frac{1}{\sqrt{2}}(|T\rangle + |R\rangle)$. In order for H to be unitary we must have

$H|R\rangle = \frac{1}{\sqrt{2}}(|T\rangle - |R\rangle)$. In matrix form in a basis $|T\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

which is unitary. Note: that this is not the only possible solution; another one would be to take

$\tilde{H}|R\rangle = \frac{1}{\sqrt{2}}(-|T\rangle + |R\rangle)$ & $\tilde{H}|T\rangle = \frac{1}{\sqrt{2}}(|T\rangle + |R\rangle)$.

which is still unitary. This would be simply another model for the semi-transparent mirror.

b) Incoming state $|\psi\rangle = \alpha|T\rangle + \beta|R\rangle$.

After the semi-transparent mirror:

$$\frac{\alpha}{\sqrt{2}}\left(|T\rangle + |R\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|T\rangle - |R\rangle\right)$$

$$= \frac{\alpha+\beta}{\sqrt{2}}|T\rangle + \frac{\alpha-\beta}{\sqrt{2}}|R\rangle .$$

After the perfect mirrors ($|T\rangle \to |R\rangle$ and $|R\rangle \to |T\rangle$).

$$\frac{\alpha+\beta}{\sqrt{2}}|R\rangle + \frac{\alpha-\beta}{\sqrt{2}}|T\rangle$$

After the second semi-transparent mirror:

$$\frac{\alpha+\beta}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|T\rangle - |R\rangle\right) + \frac{\alpha-\beta}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|T\rangle + |R\rangle\right)$$

$$= \alpha|T\rangle - \beta|R\rangle$$

Note: If the reflection at the mirrors is modelled by say $|T\rangle \to |T\rangle$ and $|R\rangle \to |R\rangle$ we would have found $\alpha|T\rangle + \beta|R\rangle$ for the final state.

c) Classically an intensity $|\alpha|^2 + |\beta|^2 = 1$ is split in two and then recombined. ~~At~~ the end it is split again in two; so D_A clicks and D_B click both with intensity $1/2$.

d) Quantum mechanically these intensities are $|\alpha|^2$ for D_A and $|\beta|^2$ for D_B.

c) After first beam splitter $|T\rangle \to \frac{1}{\sqrt{2}}(|T\rangle + \frac{i}{\sqrt{2}}|R\rangle)$

and $|R\rangle \to \frac{1}{\sqrt{2}}(i|T\rangle + |R\rangle)$ so that the

transition is unitary: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

State $\alpha|T\rangle + \beta|R\rangle$ becomes:

$$\frac{\alpha}{\sqrt{2}}(|T\rangle + i|R\rangle) + \frac{\beta}{\sqrt{2}}(i|T\rangle + |R\rangle).$$

$$= \frac{\alpha + i\beta}{\sqrt{2}}|T\rangle + \frac{i\alpha + \beta}{\sqrt{2}}|R\rangle.$$

After the mirrors:

$$\frac{\alpha + i\beta}{\sqrt{2}}|R\rangle + \frac{i\alpha + \beta}{\sqrt{2}}|T\rangle$$

After the second beam splitter:

$$\frac{\alpha + i\beta}{\sqrt{2}}\frac{1}{\sqrt{2}}(i|T\rangle + |R\rangle) + \frac{i\alpha + \beta}{\sqrt{2}}\frac{1}{\sqrt{2}}(|T\rangle + i|R\rangle).$$

$$= i\alpha|T\rangle + i\beta|R\rangle = i(\alpha|T\rangle + \beta|R\rangle).$$

Up to a phase factor $\exp(i\frac{\pi}{2})$ the entry and output states are the same.

The $\text{Prob}(D_A \text{ clicks}) = |\alpha|^2$; $\text{Prob}(D_B \text{ clicks}) = |\beta|^2$.

f) With a phase shifter $S|R\rangle = e^{i\phi}|R\rangle$ and $S|T\rangle = |T\rangle$ and the interferometer as in questions a) and b) the calculation becomes:

after beam splitter: $$\frac{\alpha+\beta}{\sqrt{2}}|T\rangle + \frac{\alpha-\beta}{\sqrt{2}}|R\rangle$$

phase shifter $$\frac{\alpha+\beta}{\sqrt{2}}|T\rangle + \frac{\alpha-\beta}{\sqrt{2}}e^{i\phi}|R\rangle$$

mirrors $$\frac{\alpha+\beta}{\sqrt{2}}|R\rangle + \frac{\alpha-\beta}{\sqrt{2}}e^{i\phi}|T\rangle$$

second beam splitter $$\frac{\alpha+\beta}{2}\left(|T\rangle + |R\rangle\right) + \frac{\alpha-\beta}{2}e^{i\phi}\left(|T\rangle - |R\rangle\right)$$

$$= \left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}e^{i\phi}\right)|T\rangle + \left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}e^{i\phi}\right)|R\rangle.$$

$$= \left\{\frac{\alpha}{2}(1+e^{i\phi}) + \frac{\beta}{2}(1-e^{i\phi})\right\}|T\rangle$$

$$+ \left\{\frac{\alpha}{2}(1-e^{i\phi}) + \frac{\beta}{2}(1+e^{i\phi})\right\}|R\rangle.$$

$$\begin{cases} \text{Prob}(D_A \text{ clicks}) = \left|\frac{\alpha}{2}(1+e^{i\phi}) + \frac{\beta}{2}(1-e^{i\phi})\right|^2 \\ \text{Prob}(D_B \text{ clicks}) = \left|\frac{\alpha}{2}(1-e^{i\phi}) + \frac{\beta}{2}(1+e^{i\phi})\right|^2. \end{cases}$$

Note: for $\phi = 0$ we recover the previous result of question d) namely $\text{Prob}(D_A \text{ clicks}) = |\alpha|^2$ and $\text{Prob}(D_B \text{ clicks}) = |\beta|^2$.

Homeworks September 26 - 2013.

Problem 1. Polarisation measurements.

a). From the Born rule

$$\text{Prob}(p_\alpha = +1) = |\langle \alpha | \theta \rangle|^2$$

$$= |(\cos\alpha \langle x| + \sin\alpha \langle y|)(\cos\theta |x\rangle + \sin\theta |y\rangle)|^2$$

$$= |\cos\alpha \cos\theta \langle x|x\rangle + \sin\alpha \sin\theta \langle y|y\rangle$$
$$+ \cos\alpha \sin\theta \langle x|y\rangle + \sin\alpha \cos\theta \langle y|x\rangle|^2$$

$$= |\cos\alpha \cos\theta + \sin\alpha \sin\theta|^2 = |\cos(\alpha - \theta)|^2.$$

Similarly

$$\text{Prob}(p_\alpha = -1) = |\langle \alpha_\perp | \theta \rangle|^2 = (\cos(\alpha_\perp - \theta))^2$$
$$= (\sin(\alpha - \theta))^2 \quad \text{since } \alpha_\perp = \alpha + \frac{\pi}{2}.$$

b) Let $P_\alpha = (+1) |\alpha\rangle\langle\alpha| + (-1) |\alpha_\perp\rangle\langle\alpha_\perp|$.

$\langle P_\alpha \rangle$ = expectation value of P_α

$$= (+1) \cos^2(\alpha - \theta) + (-1) \sin^2(\alpha - \theta)$$
$$= \cos 2(\alpha - \theta).$$

$$\langle P_\alpha^2 \rangle - \langle P_\alpha \rangle^2 = \left((+1)^2 \cos^2(\alpha - \theta) + (-1)^2 \sin^2(\alpha - \theta)\right) - (\cos 2(\alpha - \theta))^2$$
$$= 1 - (\cos 2(\alpha - \theta))^2$$

Now we check that the quantum expressions are the same!

$$\langle\theta| P_\alpha |\theta\rangle = \langle\theta| \left(|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp| \right) |\theta\rangle$$

$$= \langle\theta|\alpha\rangle\langle\alpha|\theta\rangle - \langle\theta|\alpha_\perp\rangle\langle\alpha_\perp|\theta\rangle$$

$$= |\langle\alpha|\theta\rangle|^2 - |\langle\alpha_\perp|\theta\rangle|^2$$

$$= (\cos(\alpha-\theta))^2 - (\cos(\alpha_\perp-\theta))^2$$

$$= (\cos(\alpha-\theta))^2 - (\sin(\alpha-\theta))^2$$

$$= \cos(2(\alpha-\theta)).$$

Similarly:

$$\langle\theta| P_\alpha^2 |\theta\rangle = \langle\theta| \left(\overset{(+1)^2}{|\alpha\rangle\langle\alpha|} + \overset{(-1)^2}{|\alpha_\perp\rangle\langle\alpha_\perp|} \right) |\theta\rangle$$

$$= \langle\theta|\alpha\rangle\langle\alpha|\theta\rangle + \langle\theta|\alpha_\perp\rangle\langle\alpha_\perp|\theta\rangle$$

$$= |\langle\alpha|\theta\rangle|^2 + |\langle\alpha_\perp|\theta\rangle|^2$$

$$= 1.$$

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c) The commutator $[P_\alpha, P_\beta] = P_\alpha P_\beta - P_\beta P_\alpha$ is given by

$$P_\alpha P_\beta = \left(|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp| \right)\left(|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp| \right)$$

$$= |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| - |\alpha\rangle\langle\alpha|\beta_\perp\rangle\langle\beta_\perp|$$
$$- |\alpha_\perp\rangle\langle\alpha_\perp|\beta\rangle\langle\beta| + |\alpha_\perp\rangle\langle\alpha_\perp|\beta_\perp\rangle\langle\beta_\perp|.$$

$$= \cos(\alpha-\beta)\,|\alpha\rangle\langle\beta| - \cos(\alpha-\beta_\perp)\,|\alpha\rangle\langle\beta_\perp|$$
$$- \cos(\alpha_\perp-\beta)\,|\alpha_\perp\rangle\langle\beta| + \cos(\alpha_\perp-\beta_\perp)\,|\alpha_\perp\rangle\langle\beta_\perp|.$$

If we interchange β and α we find:

$$P_\beta P_\alpha = \cos(\beta-\alpha)\, |\beta\rangle\langle\alpha| - \cos(\beta-\alpha_\perp)\, |\beta\rangle\langle\alpha_\perp|$$

$$- \cos(\beta_\perp-\alpha)\, |\beta_\perp\rangle\langle\alpha| + \cos(\beta_\perp-\alpha_\perp)\, |\beta_\perp\rangle\langle\alpha_\perp|.$$

The difference is: $[P_\alpha, P_\beta] =$:

$$\cos(\alpha-\beta)\, \big(|\alpha\rangle\langle\beta| - |\beta\rangle\langle\alpha|\big)$$

$$- \cos(\alpha-\beta_\perp)\, \big(|\alpha\rangle\langle\beta_\perp| - |\beta_\perp\rangle\langle\alpha|\big).$$

$$- \cos(\alpha_\perp-\beta)\, \big(|\alpha_\perp\rangle\langle\beta| - |\beta\rangle\langle\alpha_\perp|\big).$$

$$+ \cos(\alpha_\perp-\beta_\perp)\, \big(|\alpha_\perp\rangle\langle\beta_\perp| - |\beta_\perp\rangle\langle\alpha_\perp|\big).$$

Now we compute $\langle\theta|[P_\alpha, P_\beta]|\theta\rangle$

$$= \cos(\alpha-\beta)\, \big(\underbrace{\cos(\alpha-\theta)\cos(\beta-\theta) - \cos(\beta-\theta)\cos(\alpha-\theta)}_{0}\big)$$

$$- \cos(\alpha-\beta_\perp)\, \big(\underbrace{\cos(\alpha-\theta)\cos(\beta_\perp-\theta) - \cos(\beta_\perp-\theta)\cos(\alpha-\theta)}_{0}\big)$$

$$- \cos(\alpha_\perp-\beta)\, \big(\underbrace{\cos(\alpha_\perp-\theta)\cos(\beta-\theta) - \cos(\beta-\theta)\cos(\alpha_\perp-\theta)}_{0}\big)$$

$$+ \cos(\alpha_\perp-\beta_\perp)\, \big(\underbrace{\cos(\alpha_\perp-\theta)\cos(\beta_\perp-\theta) - \cos(\beta_\perp-\theta)\cos(\alpha_\perp-\theta)}_{0}\big).$$

$= 0.$

$$\Rightarrow \quad \langle \theta | [P_\alpha, P_\beta] | \theta \rangle = 0 .$$

and in this case the Heisenberg inequality is trivial.

- However in general $[P_\alpha, P_\beta] \neq 0$ and if we take a more general polarization state than the linear pol state $|\theta\rangle$ (say we take circular or elliptic pol) then $\langle \psi | [P_\alpha, P_\beta] | \psi \rangle$ will not necessarily vanish and the inequality becomes non-trivial.

Problem 2 : Heisenberg inequality for Hermitian Matrices

- let $A' = A - \langle\psi|A|\psi\rangle . I$, I = identity matrix.

$$B' = B - \langle\psi|B|\psi\rangle . I$$

we have $\langle\psi|A'|\psi\rangle = \langle\psi|B'|\psi\rangle = 0$.

and $\langle\psi|A^2|\psi\rangle - (\langle\psi|A|\psi\rangle)^2 = \langle\psi|A'^2|\psi\rangle$

$$\langle\psi|B^2|\psi\rangle - (\langle\psi|B|\psi\rangle)^2 = \langle\psi|B'^2|\psi\rangle.$$

- Moreover can be checked explicitely be cause are numbers.

$$[A, B] = [A - \langle\psi|A|\psi\rangle , B - \langle\psi|B|\psi\rangle]$$

$$= [A', B'] .$$

- Heisenberg's inequality is equivalent to :

$$\Delta A' . \Delta B' \geq \frac{1}{2} |\langle\psi|[A', B']|\psi\rangle|.$$

~~Consider $\langle\psi|(A'+i\lambda B')^2|\psi\rangle$. This is a polynomial~~

~~in λ~~:

~~$\langle\psi|(A'+i\lambda B')^2|\psi\rangle$~~

- Consider $P(\lambda) = \langle\psi|(A'+i\lambda B')(A' - i\lambda B')|\psi\rangle$.

$$= \lambda^2 \langle\psi|B'^2|\psi\rangle - \lambda 2i \langle\psi|[A', B']|\psi\rangle$$

$$+ \langle\psi|A'^2|\psi\rangle .$$

- Since $[A', B']^{+} = (A'B')^{+} - (B'A')^{+}$

$$= B'A' - A'B'$$

$$= [B', A'] = -[A', B']$$

we have that $(i[A', B'])^{+} = i[A', B']$

$\Rightarrow$ $i\langle\psi|[A', B']|\psi\rangle$ is real.

- So the coefficients of the polynomial are real. Since we know that $P(\lambda) \geq 0$ (because its the average of a hermitian matrix) ~~and for d ...~~ for λ real we must have

$\Delta(\lambda) \leq 0$ i.e

$$|\langle\psi|[A', B']|\psi\rangle|^2 - 4\langle\psi|B'^2|\psi\rangle\langle\psi|A'^2|\psi\rangle \leqslant 0 .$$

$$\Rightarrow \quad \Delta A' \Delta B' \geqslant \frac{1}{2}|\langle\psi|[A', B']|\psi\rangle| .$$

Problem 3: Entropic Uncertainty Principle.

See Nielsen and Chuang. page 503.
"Quantum Computation and Quantum Information".