Quantum Information and Computation

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Preface

Draft of course notes

Part I

Quantum Mechanics and Quantum Bits

1 Experiments with light

At the beginning of the 20th century a major change of paradigm occured in the laws of physics. This revolution was triggered by a host of experimental discoveries which lead to a major revision of our concepts of particle, wave and measurements of observables such as for example position, velocity, magnetic moment. The quantum theory that emerged is today the best tested theory of physical phenomena. The classical laws of physics are seen as a limiting case of quantum laws, that are valid when quantum effects can be neglected. This is the case for a wide range of phenomena which roughly speaking are macroscopic phenomena of our everyday life for which Newton's law of motion and Maxwell equations are adequate. Relativistic effects also count as classical phenomena even though it is not clear if they affect our everyday life (the GPS takes into account relativistic effects). Quantum effects cannot be neglected when we want to describe microscopic phenomena. But note that macroscopic quantum phenomena also exist and the borderline between classical and quantum behaviors is a deep, subtle and not totally solved problem. In any case, quantum theory explains the chemical bond, is thus at the basis of chemistry, it explains the stucture of the atom and the periodic table of elements, and is the basis for nuclear, particle and high energy physics. Quantum mechanics is also necessary to explain many properties of condensed matter for example metals, semi-conductors, magnets, superconductors, superfluids. Quantum mechanics is necessary to explain the interaction of matter and light.

Quantum mechanics was largely discovered by studying (experimentally) the interactions of matter with light. The early experiments of the 20th century, and some of the late 19th century, forced physicist to revise completely their views on the intimate nature of light and matter. It was gradually realized that light has both particle-like and wave-like behaviors. Similarly particles (eg. the electron) have both particle-like and wave-like behaviors. Today we view these constituents of matter as entities called "quantum fields". Wave and particle behaviors are manifestations of the quantum fields.

The laws of physics are expressed in mathematical language. It is thus not so surprising that these conceptual revolutions were couched in a mathematical formalism that departs quite radically from the one of classical physics. For example Heisenberg was bold enough to represent observable and measurable quantities, such as position and velocity of an electron orbiting an atom, by "matrices" or

"linear operators". From our modern perpective it is hard to appreciate why this was so bold. Let us just point out here that matrices were not part of the curriculum of physics students in the 1920's and that Heisenberg constructed the rule of matrix multiplication and other linear algebra facts by means of guessing at the laws of physics. He was guided by experiment, was a genius, and guessed right! The mathematical formalism of quantum mechanics has posed new interesting problems in functional analysis, geometry, group theory (today quantum information theory also offers new mathematical challenges).

The development of quantum mechanics in its modern form spans a period of at least 25-30 years between 1900 and 1930's. It is the achievement of many experimental and theoretical physicists. This was a golden age of discovery in physics full of surprising developments. It will not be possible to go through and understand the historical development of quantum mechanics in this course. Starting with chapter 2, the modern formalism of quantum mechanics is presented. This mathematical formalisation of the physical laws discovered by the founding fathers, was first clearly spelled out by von Neumann around 1940, and has remained for the main part unchanged since then.

Before proceeding directly to the mathematical formalism it is nevertheless good to motivate it through some experiments that can be performed with light. We will gradually introduce some of the basic ideas of quantum mechanics through the discussion of these experiments. This is the goal of the chapter.

1.1 Electromagnetic waves

According to Maxwell (1862) and Hertz (1886), light is an electromagnetic wave of electric $\mathbf{E}(\mathbf{x},t)$ and magnetic $\mathbf{B}(\mathbf{x},t)$ fields freely oscillating in vacuum. The solutions of Maxwell equations in empty space are superpositions of monochromatic modes of frequency ω. A mode, or plane wave, propagating along the z axis, is given by

$$\mathbf{E}(\mathbf{x},t) = Re\,\mathbf{E}_0\, e^{i(kz-\omega t)}, \qquad \mathbf{B}(\mathbf{x},t) = \frac{1}{c}\hat{\mathbf{z}}\times\mathbf{E}(\mathbf{x},t), \qquad \omega = ck \tag{1.1}$$

The amplitude vector $\mathbf{E}_0$ (thus $\mathbf{E}$ and $\mathbf{B}$ also) always belongs to the $(x,y)\perp z$ plane,

$$\mathbf{E}_0 = E_0 \begin{bmatrix} \cos\theta e^{i\delta_x} \\ \sin\theta e^{i\delta_y} \\ 0 \end{bmatrix} \tag{1.2}$$

The energy per unit time per unit surface that would be imparted to a material object by the wave, is given by the norm of the Poynting vector

$$\mathbf{S} = \epsilon_0 c^2 \mathbf{E}\times\mathbf{B} \tag{1.3}$$

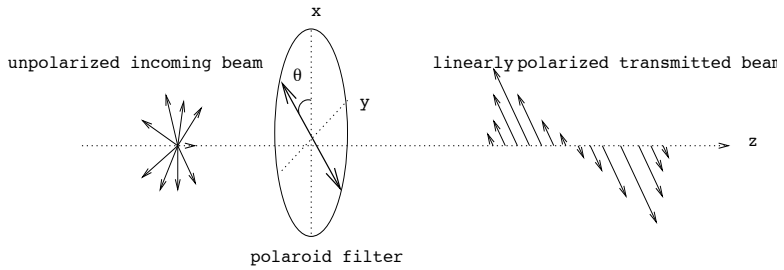

Figure 1.1 Preparation of beam polarized along θ

A convenient measure of the intensity I of the wave is given by the average of this norm, over a period $T = \frac{2\pi}{\omega}$,

$$I = \frac{1}{2}\epsilon_0 c|\mathbf{E}_0|^2 = \frac{1}{2}\epsilon_0 c E_0^2 \tag{1.4}$$

From (1.1), (1.2) it follows that the tip of the electric (and hence also magnetic) field vector describes, as a function of time, an ellipse in the (x, y) plane. There are two degenerate cases of special importance. *Linear polarization* corresponds to $\delta_x - \delta_y = m\pi$ (m integer) and the tip of the field oscillates in the (x, y) plane on a line making the angle θ with x (m even/odd). For $\theta = \frac{\pi}{4}$ (so that $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$) and $\delta_x - \delta_y = m\frac{\pi}{2}$ (m odd integer) the polarization is left/right *circular* which means that the tip of the field rotates along a circle of radius E_0.

A light beam can be easily prepared in a state of linear polarization with the help of a filter which transmits only the component of the electric field along θ. All our subsequent discussion does not rely on a detailed explanation of the phenomenon and we do not need to know more about it[1]. Such a device is called a *polarizer* with axis θ (figure 1.1).

Analyzer-detector apparatus. Assume that a source of light has been prepared in a state of linear polarization along θ as in figure 1.1.

$$\mathbf{E}_{\text{in}}(\mathbf{x}, t) = E_0 \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} Re\, e^{i(kz-\omega t)} \tag{1.5}$$

The intensity of the prepared beam (1.5) is proportional to E_0^2. Suppose now that this ray is transmitted through a second polarizer at an angle α. This second polarizer is called the *analyzer*. The light is then collected by a detector[2] and its

[1] In fact so-called absorptive polarizers are made of sheets of anisotropic crystals allowing electron motion preferentially in the $\theta_\perp$ direction. The $\theta_\perp$ component of the electric field sets electrons into a state of oscillation which produces the emission of an emitted anti-phase electromagnetic wave polarized along $\theta_\perp$. The later cancels the progressive $\theta_\perp$ component of the wave so that the net effect is to leave out a θ transmitted component and a $\theta_\perp$ reflected component.

[2] This can be a photoelectric cell which transforms the electromagnetic energy into a current.

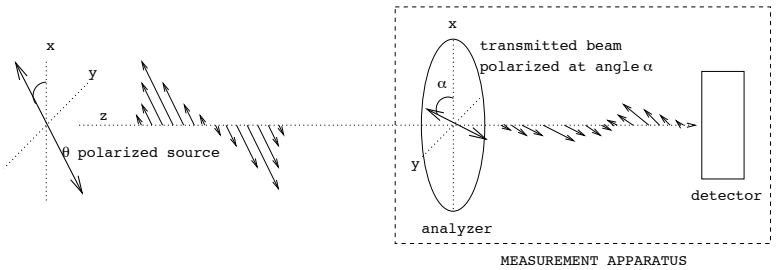

Figure 1.2 analyzer-detector measurement apparatus

intensity measured (see figure 1.2). The electric field of the final beam is obtained by projecting the incoming electric field on the analyzer axis $\mathbf{e}_\alpha$

$$\mathbf{E}_{\text{out}} = (\mathbf{E}_{\text{in}} \cdot \mathbf{e}_\alpha)\, \mathbf{e}_\alpha = E_0 \cos(\theta - \alpha) \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} Re\, e^{i(kz-\omega t)} \tag{1.6}$$

and the intensity received in D is proportional to $E_0^2 \cos^2(\theta - \alpha)$. To summarize, when a beam polarized along θ is transmitted through an analyzer at an angle α, the outgoing beam is polarized along α and the fraction of intensity collected by the detector (average power per unit surface) is[3]

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \cos^2(\theta - \alpha) \tag{1.7}$$

In particular if $\alpha - \theta = 0, \pi$ all the light passes through the analyzer, while if $\alpha - \theta = \pm\frac{\pi}{2}$ none of it is transmitted. The analyzer-detector system can be used as a measurement apparatus to determine the polarization of a wave (assuming we know a priori that it is linear) by adjusting the angle α such that the collected intensity varies from 0 to its maximal value. Let us now describe two simple experiments with electromagnetic waves.

Polarizing beam-splitter experiment. There exist prisms[4] that have the property of splitting a beam in two linearly polarized ones, one is polarized perpendicular to the incidence plane while the other is polarized parallel to that plane. In figure 1.3 the incidence plane is (x, z) so one ray has y polarization while the other one has x polarization. Two detectors D_x and D_y measure the outgoing intensities of each beam. Note that the polarization degree of freedom is coupled to the orbital (path of ray) degree of freedom. Before the polarizing beam-splitter the electric filed is given by (1.5) and has intensity proportional

[3] Malus law.

[4] These are made of quartz or calcite crystals whose refraction index are different for polarization perpendicular to, versus into, the incidence plane. Such crystals are called birefringent, one ray is called ordinary because the direction of refraction obeys the usual Snell law, while the other ray is called extraordinary.

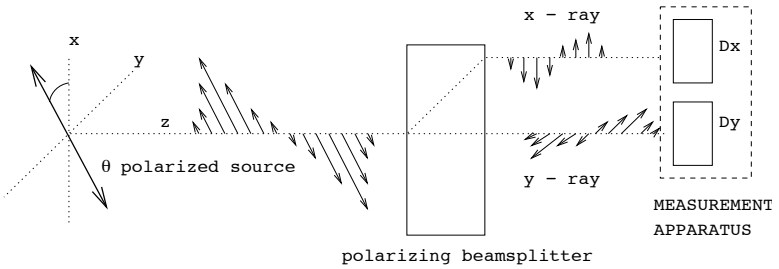

Figure 1.3 polarizing beam-splitter experiment

to E_0^2. After the beam-splitter the x-polarized ray has an electric field

$$\mathbf{E}_x = E_0 \begin{bmatrix} \cos\theta \\ 0 \\ 0 \end{bmatrix} Re\, e^{i(kz-\omega t)} \tag{1.8}$$

and the intensity detected at D_x is proportional to $E_0^2 \cos^2\theta$, while the y-polarized ray has a field

$$\mathbf{E}_y = E_0 \begin{bmatrix} 0 \\ \sin\theta \\ 0 \end{bmatrix} Re\, e^{i(kz-\omega t)} \tag{1.9}$$

and its intensity measured by D_y is proportional to $E_0^2 \sin^2\theta$. *Both detectors collect a fraction of the intensity*,

$$\frac{I_{\text{out},x}}{I_{\text{in}}} = \cos^2\theta, \qquad \frac{I_{\text{out},y}}{I_{\text{in}}} = \sin^2\theta \tag{1.10}$$

In this experiment absorption and reflection by the prism are negligible so that the sum of the these two fractions equals 1.

Decomposition-recombination experiment. Once we have decomposed light with a polarizing beam-splitter, we can recombine it with a symmetric prism. We analyze the recombined beam with an analyzer-detector apparatus (see figure 1.4). Let us carefully review the situation. Before the first beam-splitter we have *one ray* with electric field given by (1.5). The first beam-splitter splits the ray in *two parts* with electric fields given by (1.8) and (1.9). After the second beam-splitter the two rays *interfere* and the electric field of the recombined beam is the sum of (1.8) and (1.9), which equals (1.5). *The fraction of intensity collected by the analyzer-detector system is*

$$\cos^2(\theta - \alpha) \tag{1.11}$$

a fact consistent with the first experiment.

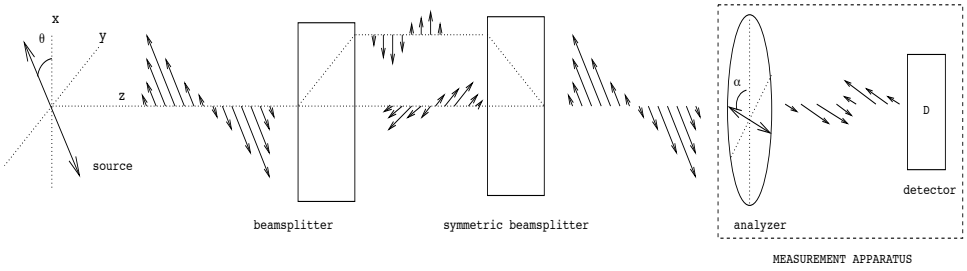

Figure 1.4 decomposition-recombination experiment

1.2 Photons

The works of Planck (1900) on the spectrum of black-body radiation, of Einstein (1905) on the photoelectric effect and Bohr (1913) on the atomic structure (and spectral lines), taught us that the interaction of light with matter occurs through discrete quanta (quantities) of energy and momentum that are absorbed and emitted. These quanta are called photons, and each photon carries an energy $\hbar\omega$ and momentum $\hbar k$ (where $\omega = ck$ still holds). If we think of the beam as a collection of independent photons, its intensity is $\hbar\omega c\frac{N}{V}$ where $\frac{N}{V}$ is the number of photons per unit volume[5]. Identifying this quantity with (1.4) we find a relation between the electric field and the number of photons associated to the electromagnetic wave.

If we diminish sufficiently the intensity of the source we arrive at a situation where in principle photons are emitted one by one. We will repeat the experiments with such a *single photon source*, that prepares them in a state of polarization θ.

Analyzer-detector apparatus. Let us first discuss how the analyzer-detector measurement apparatus works. We repeat the experiment of figure 1.2 and collect photons at the detector D. When a photon hits the detector the later clicks (an electric pulse is triggered) - we record this event as a 1, otherwise we record 0. This experiment produces a sequence

$$100111100010101001110 1... \tag{1.12}$$

that looks random and where the empirical fraction of 1's is $\cos^2(\theta - \alpha)$. From this experiment we infer

$$\text{probability of detecting a photon} = \cos^2(\theta - \alpha) \tag{1.13}$$

In particular if $\alpha - \theta = 0, \pi$ all photons are detected while if $\alpha - \theta = \pm\frac{\pi}{2}$ no photon is detected.

This experiment suggests that *photons behave as particles which carry a polarization degree of freedom.* Indeed if they would behave as waves, then a part

[5] $c\frac{N}{V}$ is the number of photons per unit time per unit surface that hit a detector.

of the wave would be transmitted through the analyzer and some energy would always be measured in the detector. However *the event is discrete, the detector clicks or does not click.* Moreover *it seems impossible to predict the precise polarization outcome for each individual photon: clicks are random.* Note that the statistics of the outcomes seems to satisfy a definite formula (1.13); and this formula is the one found in the theory of electromagnetic waves (!).

The randomness of the outcome is a fundamental feature of the measurement process for quantum systems and that it is not at all obvious to reconcile this fact with our classical intuitions. One could attempt a classical interpretation[6] by saying that the photon is a particle-like object that undergoes complicated but otherwise deterministic collision processes within the analyzer, which result in a probability $\cos^2(\theta - \alpha)$ of being transmitted. Such attempts do not resist the tests of other experiments.

Let us now repeat the two previous experiments with photons that are sent one by one.

Polarizing beam-splitter experiment. Each single photon (polarized at an angle θ) goes through the prism. We observe that either D_x clicks (the upper detector register a 1 and the lower a 0) or D_y clicks (the upper detector registers a 0 and the upper a 1); but they never click simultaneously. We record two random complementary sequences with respective fractions of 1 equal to $\cos^2\theta$ and $\sin^2\theta$. Empirically,

$$\text{prob detect photon at } \mathrm{D_y} = \sin^2\theta, \qquad \text{prob detect photon at } \mathrm{D_x} = \cos^2\theta \quad (1.14)$$

The sum is equal to one which means that the photon has certainly passed through the beam-splitter.

The fact that the detectors never click simultaneously suggest as above that the *photons behave as particles.* Indeed, would they behave as waves, both detectors would collect some energy.

One may attempt the same (wrong) classical interpretation as above. A photon is a *particle*, which due to complicated but otherwise deterministic collisions with the crystal, is deflected towards the lower path with probability $\sin^2\theta$ *or* through the upper path with probability $\cos^2\theta$. This turns out to be incompatible with the next experiment.

Decomposition-recombination experiment. let us consider again the setting of figure 1.4. When photons are sent one by one we again record a sequence of random clicks, and we infer from this sequence

$$\text{prob detect photon at D} = \cos^2(\theta - \alpha) \qquad (1.15)$$

This should comes as a great surprise to the reader. Indeed this result *is not consistent with the particle-like picture of a photon, but rather with a wave-like picture*, as we now show.

[6] in the spirit of statistical mechanics, say

Theoretical prediction of the particle picture. If a photon takes the lower path in figure 1.4 its polarization is horizontal before the second beam-splitter and comes out of it in a horizontal state. Therefore the probability of transmission of such a lower-path photon through the analyzer is $\cos^2(\frac{\pi}{2} - \alpha) = \sin^2 \alpha$. Therefore

$$\text{prob(D clicks} \mid \text{lower path)} = \sin^2 \alpha \tag{1.16}$$

If the photon takes the upper path its polarization is vertical just before the second beam-splitter and comes out in a state of vertical polarization. Therefore the probability of transmission of such an upper-path photon is $\cos^2(0 - \alpha)$ and

$$\text{prob(D clicks} \mid \text{upper path)} = \cos^2 \alpha \tag{1.17}$$

Now, we have

$$\begin{aligned} \text{prob(D clicks)} =& \text{prob(D clicks} \mid \text{lower path)prob(lower path)} \\ &+ \text{prob(D clicks} \mid \text{upper path)prob(upper path)} \end{aligned} \tag{1.18}$$

Thus because of (1.14), (1.16), (1.17)

$$\text{prob detect photon at D} = \sin^2 \theta \sin^2 \alpha + \cos^2 \theta \cos^2 \alpha \tag{1.19}$$

This contradicts the experimental result (1.15) *and is therefore plain wrong !*

The term that is missing is precisely

$$2 \cos\theta \cos\alpha \sin\theta \sin\alpha \tag{1.20}$$

which, in wave theory, appears because of the *interference* between the x and y components of the electric field. This suggests that a single photon follows both paths, just as a wave would do, and interferes with itself just as a wave would do[7].

Let us summarize. We face the following situation: the decomposition experiment suggests that photons behaves in a *particle-like* manner, while he recombination experiment (1.4) suggests that photons behave in a *wave-like* fashion. As for most dilemmas, the resolution offered by *quantum theory teaches us that both pictures are two faces of a more subtle reality that goes beyond this dichotomy.* One sometimes refers to this dual behavior of light, and all known forms of matter, as the "particle-wave duality" or the "complementarity principle".

[7] The mathematical formulation of this picture leads to the Feynman formulation of quantum mechanics in terms of path integrals (1948). In this course we will adopt the Dirac and von Neumann formulation in terms of Hilbert spaces (1932). We stress that these formulations have essentially nothing new to say about the measurement postulate (formulated later).

1.3 The quantum setting: first encounter

In fact all known forms of matter[8] display this particle/wave duality. As we will now see quantum mechanics offers us a picture which accommodates both behaviors and supersedes the classical pictures of wave and particle[9].

We will illustrate how the rules of quantum theory consistently explain the three experiments. The situation will be modeled in the simplest possible way which retains the basic essence of quantum mechanics.

The state of a photon is described by two degrees of freedom, an *orbital* degree of freedom and a *polarization* degree of freedom. Let us first concentrate on polarization. The state of polarization is described by a unit vector $\mathbf{e}$ perpendicular to the direction of motion. Following Dirac we call these state vectors *kets* and denote them as $|\mathbf{e}\rangle$. Since the polarization vector lies in the x, y plane it can be described in a orthonormal basis $|\updownarrow\rangle$, $|\leftrightarrow\rangle$, corresponding to the two linear states of polarization along x and y

$$|\mathbf{e}\rangle = \lambda|\updownarrow\rangle + \mu|\leftrightarrow\rangle, \qquad |\lambda|^{\mathbf{2}} + |\mu|^{\mathbf{2}} = \mathbf{1} \tag{1.21}$$

Here λ and μ are complex numbers. thus a general polarization state is a normalized two component vector belonging to $\mathbf{C}^2$. The space $\mathbf{C}^2$ is our first and simplest example of a space of quantum states.

A state of linear polarization along θ corresponds to $\lambda = \cos\theta$ and $\mu = \sin\theta$, so that (1.29) becomes

$$|\theta\rangle = \cos\theta|\updownarrow\rangle + \sin\theta|\leftrightarrow\rangle \tag{1.22}$$

On the other hand for circular polarization the x and y components of the polarization vector have a $\frac{\pi}{2}$- phase difference. Two basis states with circular polarization are,

$$|L/R\rangle = \frac{1}{\sqrt{2}}(|\updownarrow\rangle \pm i|\leftrightarrow\rangle) \tag{1.23}$$

Given a state vector $|\Phi\rangle$ its *adjoint* (also called hermitian conjugate) is obtained by taking the complex conjugate and transposing $\overline{|\Phi\rangle}^T$. This is denoted as a *bra*

$$\langle\Phi| = \overline{|\Phi\rangle}^T \tag{1.24}$$

The usual inner product (defined over a complex vector space) is called the *bracket*

$$\langle\Psi|\Phi\rangle = (\overline{|\Psi\rangle}^T) \cdot (|\Phi\rangle) \tag{1.25}$$

[8] For example photons, electrons, nuclei and their constituents ...

[9] According to modern physics, matter is described by relativistic quantum fields. There are underlying quantum fields (e.g. the quantum electromagnetic field, the quantum electronic field, the quark field etc...) which may manifest themselves in a wave-like or particle-like fashion depending on the situation. We will not introduce field theoretical notions in this course.

As an example consider the inner product between two polarization state vectors. First the conjugate of a linearly polarized state is

$$\langle\alpha| = \langle\updownarrow| \cos\alpha + \langle\leftrightarrow| \sin\alpha \tag{1.26}$$

The inner product with $|\theta\rangle$ then is

$$\begin{aligned}\langle\alpha \mid \theta\rangle &= (\langle\updownarrow| \cos\alpha + \langle\leftrightarrow| \sin\alpha) \cdot (\cos\theta |\updownarrow\rangle + \sin\theta |\leftrightarrow\rangle) \\ &= \cos\alpha\cos\theta + \sin\alpha\sin\theta \\ &= \cos(\theta - \alpha)\end{aligned} \tag{1.27}$$

To obtain the second equality one expands the braces into four terms, uses linearity of the bracket and the orthonormality condition,

$$\langle p \mid p'\rangle = \delta_{pp'} \tag{1.28}$$

This trivial calculation has been done in the linear polarization orthonormal basis $\{|\leftrightarrow\rangle, |\updownarrow\rangle\}$. It is instructive to check that the circularly polarized states $\{|L\rangle, |R\rangle\}$ form another orthonormal basis of the two dimensional complex vector space.

Let us now introduce the orbital degree of freedom in the picture. For a freely moving photon, i.e a photon that does not interact with a material object, the orbital state is entirely described once we know its *momentum* $\mathbf{k}$, which has a *direction* and a *norm* $k = \frac{\omega}{c}$. The state vector is now denoted as $|\mathbf{k}, \mathbf{e}\rangle$. This state freely evolves with time and for a photon of frequency ω the time evolution simply amounts to a multiplicative phase factor, which does not change the momentum and the polarization. The photon state at time t is

$$|\Psi_{\mathbf{k},\mathbf{e}}(t)\rangle = e^{-i\omega t}|\mathbf{k}, \mathbf{e}\rangle \tag{1.29}$$

An explanation is in order here about the kets indexed by two degrees of freedom. We will see in the next chapter that the mathematical rule to combine degrees of freedom is the tensor product; this means that $|\mathbf{k}, \mathbf{e}\rangle = |\mathbf{k}\rangle \otimes |\mathbf{e}\rangle$ and that the inner product is

$$\langle\mathbf{k}', \mathbf{e}' \mid \mathbf{k}, \mathbf{e}\rangle = \langle\mathbf{k}' \mid \mathbf{k}\rangle \cdot \langle\mathbf{e}' \mid \mathbf{e}\rangle \tag{1.30}$$

Finally the momentum vectors themselves form an orthonormal basis $\langle\mathbf{k}' \mid \mathbf{k}\rangle = \delta_{\mathbf{k}',\mathbf{k}}$.

As we will see in the next chapter, in general, the time-evolution of isolated systems is given by a unitary transformation. In (1.29) the unitary transformation is simply the multiplication by the phase factor. When the photon interacts with matter (for example with the analyzer, the beam-splitter) one has in principle to describe the unitary evolution of the total system (photon + analyzer or photon + beam-splitter), which is then more complicated. Here we do not have to discuss such issues as we consider only the in-going and out-going states which are those of freely moving photons.

When we make a measurement on a system, the system that is observed cannot be considered as isolated and the state is modified in a non-unitary way.

Figure 1.5 measurement with initial state $|\Psi\rangle$ and outcome $|\Phi\rangle$.

Explaining the measurement process is a subject that has been (and sometimes is still) much debated since the early days of quantum mechanics. An operational rule, to determine the outcome of a measurement is given by the so-called *measurement postulate* (Born, heisenberg, Bohr 1924-1927) in the form advocated by what has been named the Copenhagen School[10] (figure 1.5). Here we give it in a rough form, and will be more precise in the next chapter.

If a system is initially prepared in the state $|\Psi\rangle$ and the outcome of the measurement is a state $|\Phi\rangle$, the probability of the transition $|\Psi\rangle \to |\Phi\rangle$ is

$$\text{Prob}(|\Psi\rangle \to |\Phi\rangle) = |\langle\Phi \mid \Psi\rangle|^2 \tag{1.31}$$

One cannot predict the outcome of the transition but only its frequency of occurrence during repeated identical experiments with identical initial states.

The transition between the initial and final state is also called "reduction" or "collapse" of the state. In a more precise formulation of the measurement postulate, in the next chapter, we will see that the transition probabilities of all possible outcomes sum to one.

The re-interpretation of the experiments in the next section should make this rather abstract postulate a bit more "natural".

1.4 Quantum interpretation of experiments.

Analyzer-detector apparatus. We assume that the source prepares single photons in the linearly polarized, freely moving state

$$|\Psi_{\mathbf{k},\theta}(t)\rangle = e^{-i\omega t}|\mathbf{k},\theta\rangle \tag{1.32}$$

If the measurement apparatus is the analyzer-detector system of figure 1.2, the measurement postulate tells us that the probability to find the photon in state $|\mathbf{k},\alpha\rangle$ is

$$|\langle\mathbf{k},\alpha \mid \Psi_{\mathbf{k},\theta}(t)\rangle|^2 = |\langle\alpha \mid \theta\rangle|^2 = \cos^2(\theta-\alpha) \tag{1.33}$$

[10] Einstein never agreed that this rule is the final story. In his words "I, at any rate, am convinced that He (God) does not throw dice". Bohr replied "Einstein, don't tell God what to do". In any case, this rule has not been challenged by experiment so far, and there is hardly any more satisfying theoretical framework to date. In this course we stick to this rule !

This is consistent with the experimentally measured frequency of clicks in D.

Polarizing beam-splitter experiment. Before the beam-splitter the photon state is (1.32), which is equal to

$$e^{-i\omega t}(\cos\theta|\mathbf{k},\updownarrow\rangle + \sin\theta|\mathbf{k},\leftrightarrow\rangle) \tag{1.34}$$

After the beam-splitter it becomes

$$e^{-i\omega t}(\cos\theta|\mathbf{k}_u,\updownarrow\rangle + \sin\theta|\mathbf{k}_l,\leftrightarrow\rangle) \tag{1.35}$$

where $\mathbf{k}_u$ and $\mathbf{k}_l$ label the upper and lower paths[11]. Notice that contrary to (1.34), in (1.35) we cannot separate the orbital and polarization degrees of freedom into a tensor product: it can be shown that for (1.35) this is an intrinsic property that does not depend on the basis. We say that the orbital and polarization degrees of freedom have been *entangled* by the beam-splitter. Entangled states depart fundamentally from the classical picture and retain quantum correlations that are missing in the classical interpretation. As we will see in this course they play a very important role in quantum information and computation because they may offer resources that are non-classical.

Now we consider the two detectors as our measurement apparatus. The measurement postulate tells us that the probability to observe the photon in state $|\mathbf{k_u},\updownarrow\rangle$ is

$$|\langle\mathbf{k_u},\updownarrow|\ e^{-i\omega t}(\cos\theta|\mathbf{k}_u,\updownarrow\rangle + \sin\theta|\mathbf{k}_l,\leftrightarrow\rangle)|^2 = \cos^2\theta \tag{1.36}$$

Similarly the probability to observe it in the state $|\mathbf{k}_l,\leftrightarrow\rangle$ is

$$|\langle\mathbf{k}_l,\leftrightarrow|\ e^{-i\omega t}(\cos\theta|\mathbf{k}_u,\updownarrow\rangle + \sin\theta|\mathbf{k}_l,\leftrightarrow\rangle)|^2 = \sin^2\theta \tag{1.37}$$

This is consistent with the experimental fractions of clicks at D_x and D_y.

Recombination experiment. The second polarizing beam-splitter transforms the entangled state (1.35) back to (1.32). The later state enters the measurement apparatus constituted by the analyzer-detector system. Therefore the probability of observing $|\mathbf{k},\alpha\rangle$ is simply given by (1.33). This is the experimental frequency of clicks at D ! The quantum interpretation does not loose track of the interference term (1.20).

1.5 Notion of quantum bit

There exist many quantum systems in nature that can be described by state vectors which belong to the vector space $\mathbf{C}^2$, the two dimensional complex vector

[11] Here we may imagine that the paths are not quite in the same direction so that these two labels are different. In principle one should make a more complete description of the orbital part of the state that takes into account the finite width of the beams.

space. If we call $|0\rangle$ and $|1\rangle$ two orthonormal basis states a general state vector takes the form

$$|\psi\rangle = \lambda|0\rangle + \mu|1\rangle, \qquad |\lambda|^2 + |\mu|^2 = 1 \tag{1.38}$$

It will often be convenient to identify

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.39}$$

and

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.40}$$

and in quantum information theory it is customary to call this canonical basis the *computational basis*. Of course one can represent the quantum bit $|\psi\rangle$ in any other basis, and one that we will often use one that is obtained by a standard 45 degree real rotation

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{1.41}$$

This basis will be called the Hadamard basis. Since the vector space is complex we can make more general unitary transformations. For example

$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad |R\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \tag{1.42}$$

We have already seen a physical realization of a quantum bit, namely the photon polarization. If we identify the computational basis with horizontal/vertical polarized photon states, then the Hadamard basis corresponds to polarized states at 45 degree angle, and the last basis obtained by a unitary transformation is physically realized by circularly left/right polarized photons. A physically meaningful parametrization of general polarization state is

$$|\psi\rangle = e^{i\delta_x}\cos\theta|\updownarrow\rangle + e^{i\delta_y}\sin\theta|\leftrightarrow\rangle \tag{1.43}$$

If we rotate our reference frame (around z) by angle β, then the state vector is obtained from the above expression by $\theta \to \theta - \beta$. In particular if the reference frame is rotated by 2π we recover the same state vector. These states form ratgher trivial representations of the group of two-dimensional rotations (about the z-axis say).

Another very common but physically different quantum bit is the *spin* $\frac{1}{2}$. The most famous elementary particle (of obvious importance in our everyday life since it transports electricity, interacts with sunlight ...) that has spin $\frac{1}{2}$ is the electron[12]. There exist also many composite systems, such as nuclei or atoms that carry a total spin of $\frac{1}{2}$. A very rough intuitive way of thinking about spin is to view the particle (the electron say) as having intrinsic spinning motion. If

[12] Constituents of nuclei, protons and neutrons also have spin $\frac{1}{2}$. In particular the interaction of the nuclear spins with magnetic fields is at the basis of Nuclear Magnetic Resonance, used for example in medical imaging.

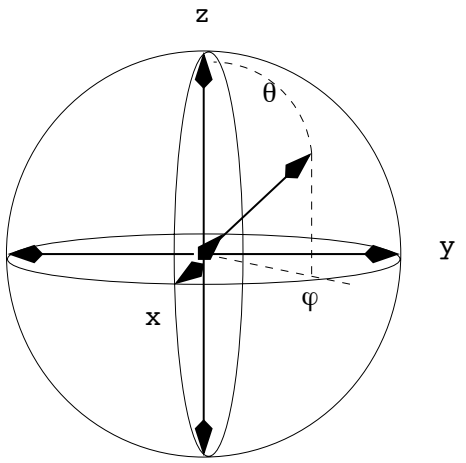

Figure 1.6 Bloch sphere. Computational (z), Hadamard (x), circular (y) basis states

the particle spins about the z axis, its spin is (pointing) $|\uparrow\rangle$ or $|\downarrow\rangle$ according to its direction of rotation. These two states form a basis and the most general spin state is

$$|\psi\rangle = \lambda|\uparrow\rangle + \mu|\downarrow\rangle, \qquad |\lambda|^2 + |\mu|^2 = 1 \tag{1.44}$$

Spin $\frac{1}{2}$ states are two dimensional (complex) representations of the group of rotations in three dimensions. A meaningful parametrization of the states is

$$|\psi\rangle = e^{i\frac{\phi}{2}} \cos\frac{\theta}{2}|\uparrow\rangle + e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2}|\downarrow\rangle \tag{1.45}$$

These states can be represented by the tip of a vector on the *Bloch sphere* (figure 1.6) with the usual spherical coordinates (θ, ϕ). We have the following correspondence (up to phase factors):

$$\theta = 0,\, \pi \qquad |\uparrow\rangle, |\downarrow\rangle, \qquad \text{particle spin along z} \tag{1.46}$$

$$\theta = \frac{\pi}{2}, \phi = 0,\, \pi \qquad |\uparrow\rangle \pm |\downarrow\rangle, \qquad \text{particle spin along x} \tag{1.47}$$

$$\theta = \frac{\pi}{2}, \phi = \pm\frac{\pi}{2} \qquad |\uparrow\rangle \pm i|\downarrow\rangle, \qquad \text{particle spin along y} \tag{1.48}$$

The polarization and spin $\frac{1}{2}$ quantum bits are different representations of the rotation group in quantum mechanics (ultimately coming from the representation of the Lorentz group of relativity).

There exist also other realizations of the quantum bit that have nothing to do with the representations of the rotation group in quantum physics. An example is given by the benzene molecule C_6H_6 that can be in the two states that differ in the arrangement of single and double electronic bonds (figure 1.7). But the

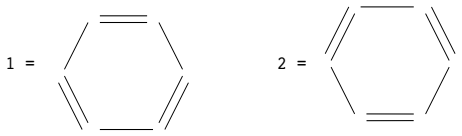

Figure 1.7 possible arrangements of chemical bonds

molecule can also be found in a *resonating state* such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \tag{1.49}$$

What is the difference between a classical bit and a quantum bit ? A classical bit is an abstraction of a physical quantity that can be reasonably well described by a two valued quantity. Examples are the charge in a capacitor, a voltage difference, or the magnetization of a Weiss domain. Classical information theory is sufficiently universal so that it does not have to account for the detailed physical properties of the classical bits. The only underlying assumption is that these exist in two definite values 0 or 1 (let us pretend that noise is absent). Suppose a classical bit is given to you and that you have no information whatsoever about its value. To gain information about its value you can observe it (measure the charge, the voltage difference) and its value is then discovered. By *discovered* we mean that it already had the observed value before the measurement, and that the measurement has not destroyed it. In this sense the information that we gain, or the uncertainty that is removed, is by definition 1 bit.

A quantum bit is also an abstraction of physical quantities as the above examples have shown. It is well described by a two dimensional complex vector. In the same spirit than in the classical case, quantum information theory is sufficiently universal so that many of its aspects are independent of the concrete physical realization. However the point is that it takes into account the general underlying laws of quantum mechanics. This means in particular that extracting information from quantum bits is quite different than in the classical case. Suppose that a quantum bit is given to you in some state $|\psi\rangle$ on which you do not have any information whatsoever. In order to determine $|\psi\rangle$, we have to observe it (agree ?). To perform a measurement we have to select an apparatus, in other words an orthonormal basis $\{|b_1\rangle, |b_2\rangle\}$. The measurement process then reduces the quantum bit to $|b_1\rangle$ or to $|b_2\rangle$. So we have lost the original state (forever) and have not gained any information because the final state depends on *our own* choice of basis. We will see later in this course, that the quantum analog of Shannon entropy, the von Neumann entropy, associated to a state like $|\psi\rangle$ is zero. Note however that if we are given many copies of $|\psi\rangle$ we can measure all of them in the same basis and get a hold of the probabilities $|\langle b_1 \mid \psi\rangle|^2$, $|\langle b_2 \mid \psi\rangle|^2$.

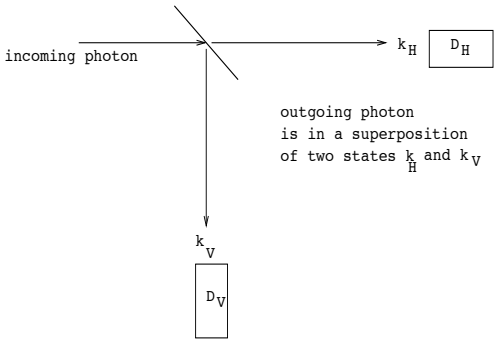

Figure 1.8 semi-transparent mirror

1.6 A random number generator

At this point the reader may well wonder if quantum laws offer any useful resource in order to process information. In this course we will see that this is so. Here we illustrate this with a very simplified model for a random number generator.

A source sends a beam of photons on a semi-transparent mirror (figure 1.8). The later splits the beam in two parts, the transmitted and reflected beams. If the source is classical we observe that the two detectors each collect a fraction of the incoming intensity of the beam. Assuming that the semi-transparent mirror is perfect each detector collects half of the intensity.

When the intensity of our source is lowered sufficiently so that it becomes a single photon source. Photons go through the mirror one at a time, we observe that *either* D_H *or* D_V clicks, never the two at the same time. We obtain a sequence of clicks 0100011101010100011011110 0 that looks Bernoulli with parameter $p = \frac{1}{2}$.

The interpretation of this experimental setup, in the framework of quantum mechanics, is as follows. We drop the polarization index as it plays no role here. A single photon is incoming in the semi-transparent mirror and the state of the photon after the mirror is,

$$e^{i\omega t}\frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle) \tag{1.50}$$

This state is a superposition. The outcome of the measurement by the detectors cannot be predicted. The probability that the photon is observed in state $|\mathbf{k}_H\rangle$ is

$$|\langle \mathbf{k}_H \mid e^{i\omega t}\frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle)|^2 = \frac{1}{2} \tag{1.51}$$

and similarly the probability that it is observed in state $\mathbf{k}_V$ is

$$|\langle \mathbf{k}_V \mid e^{i\omega t}\frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle)|^2 = \frac{1}{2} \tag{1.52}$$

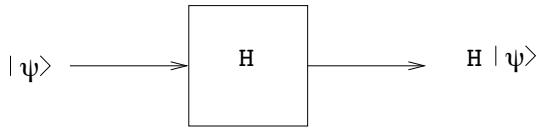

Figure 1.9 Hadamard gate as a model for a semi-transparent mirror

So the measurement process produces a perfectly random sequence.

What do we mean by "perfectly random sequence"? Of course, the sequence is perfectly random only in principle, because in the real experiment there are imperfections, for example, the source is only approximately a single photon source and the semi-transparent mirror has a small bias etc.... But the point here is that, according to the standard interpretation of quantum mechanics, the measurement process produces "true randomness" and not "pseudo-randomness": the clicks are not the result of some underlying deterministic process. This point has been much debated by the founding fathers of 20-th century physics and notably by Einstein and Bohr. According to Einstein "God does not play dice", a view that Bohr dismissed. Until today, no other theoretical framework has, successfully described as many phenomena as quantum theory does, and we have so far no experiment that forces us to abandon the standard quantum framework. It is in this sense that we declare the sequence perfectly random.

A slightly more abstract representation in quantum information theory language of this experiment is depicted on figure 1.9. We prepare and measure states in the computational basis $|0\rangle, |1\rangle$. The initial state $|0\rangle$ goes through a Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.53}$$

which produces the state

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1.54}$$

When we perform a measurement on (1.54) the outcome is $|0\rangle$ with probability

$$|\langle 0|H|0\rangle|^2 = \frac{1}{2} \tag{1.55}$$

or $|1\rangle$ with probability

$$|\langle 1|H|0\rangle|^2 = \frac{1}{2} \tag{1.56}$$

We note that quantum random number generators based on these principles have been realized and are even commercialized. See for example http://www.idquantique.com/true-random-number-generator/products-overview.html

Part II

Quantum Information Theory

Part III

Quantum Computation

Notes