Homework 4. October 31, 2013. Quantum information theory and computation

Problem 1. Bell inequality for a non-maximally entangled state.

Calculate the QM prediction for the correlation coefficient X of the CHSH quantity (see chap 4 of lecture notes) when the pair of particles is produced in the state (α real)

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1-\alpha^2)^{1/2}|11\rangle$$

For this, proceed similarly to the notes and show that the maximal value of X is $2[1+4\alpha^2(1-\alpha^2)]^{1/2}$. In this sense we can say that $\alpha = \frac{1}{\sqrt{2}}$ corresponds to a maximally entangled state (the Bell state).

Problem 2. Tsirelson inequality and maximal violation of Bell's inequality

The purpose of the exercise is to show that the set up described in the course yields the maximum possible violation of the Bell inequality.

The three 2×2 matrices X, Y, Z are called Pauli matrices. In Dirac notation they are $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$. In physics the standard notation for these matrices is σ_x, σ_y and σ_z.

It is often convenient to introduce the "vector" $\sigma = (X, Y, Z)$. For electrons this has the physical meaning of the "spin of the electron". For photons it simply corresponds to three different polarization observables: linear (45 degrees), circular and linear (0 degree).

a) Check the commutation relations $[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$.

b) Let $\mathbf{q}$ and $\mathbf{r}$ two 3-vectors. Let $Q = \mathbf{q}\cdot\sigma$ and $R = \mathbf{r}\cdot\sigma$. Check $[Q, R] = 2i(\mathbf{q}\times\mathbf{r})\cdot\sigma$

c) Let also $S = \mathbf{q}\cdot\sigma$ and $T = \mathbf{t}\cdot\sigma$. Prove the identity

$$R\otimes S + R\otimes S + R\otimes T - Q\otimes T = 4I + [Q, R]\otimes[S, T]$$

and deduce that for any state $|\psi\rangle$ of $\mathbf{C}^2\otimes\mathbf{C}^2$ we have the inequality

$$\langle\psi|R\otimes S + R\otimes S + R\otimes T - Q\otimes T|\psi\rangle \leq 2\sqrt{2}$$

d) What is the maximal value of the left hand side for a tensor product state ? Give an example of $|\psi\rangle$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ that attains the upper bound.

Problem 3. GHZ states and "local hidden variable theories"

The goal of this exercise is to discuss a thought experiment that proves that QM results cannot be replaced by local hidden variable theories, in an even stronger sense than the Bell inequality violation. In the Bell inequality set-up Alice and Bob do many measurements and compute an empirical correlation. Here there are three parties (Alice, Bob and Charlie) which do only four measurements and multiply out their result.

Consider the Green-Horne-Zeilinger state of three spins $|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle_{ABC} - |\downarrow\downarrow\downarrow\rangle_{ABC})$ where A, B, C are distant locations (which do not communicate). Consider the three observables X, Y, Z represented by the three Pauli matrices (actually we will not use Z so forget about it).

a) Show that $|GHZ\rangle$ is an eigenstate of the operators $Y_A \otimes Y_B \otimes X_C$, $Y_A \otimes X_B \otimes Y_C$, $X_A \otimes Y_B \otimes Y_C$ with eigenvalue 1. Furthermore show that $|GHZ\rangle$ is an eigenstate of $X_A \otimes X_B \otimes X_C$ with eigenvalue -1.

b) Now imagine Alice, Bob and Charlie in their labs at locations A, B and C measure the observables X and Y on their respective particles. They do the four experiments (each time on a new GHZ state):

- experiment one: Alice measures Y, Bob Y and Charlie X.
- experiment two: Alice measures Y, Bob X and Charlie Y.
- experiment three: Alice measures X, Bob Y and Charlie Y.
- experiment four: Alice measures X, Bob X and Charlie X.

From the previous question, deduce the three-particle state (i.e in $\mathbf{C}^2$) after the measurement, the value of the four observables $Y_A \otimes Y_B \otimes X_C$, $Y_A \otimes X_B \otimes Y_C$, $X_A \otimes Y_B \otimes Y_C$, $X_A \otimes X_B \otimes X_C$, and the associated probabilities of the outcomes. *Hint* : you should obtain the result without calculation !

c) Suppose now that the outcome of a measurement can be described by a local hidden variable theory. In other words suppose that Alice, Bob and Charlie have some way of computing the outcome of their experiments by functions $F_A(W, \Lambda)$, $F_B(W, \Lambda)$, $F_C(W, \Lambda)$ where the first variable W is the measurement basis (or apparatus) used i.e $W = X$ or Y and the second variable Λ is the "hidden variable" of the theory. Show that this setting is not compatible with the QM results of the four previous experiments.

Hint : there is no big calculation, you only have to multiply plus and minus ones ! You should just realize that the functions F_A, F_B and F_C take values in $\{-1, +1\}$ because these are the values taken by the observables Y and X.

Problem 4. Entanglement swapping again

Consider three close-by locations A, B, C (for example three points in *your lab*) and three very distant locations A', B', C'. Suppose we have created three entangled pairs between AA', BB', CC' in the state

$$\frac{1}{\sqrt{2}}(|00\rangle_{AA'} + |11\rangle_{AA'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{BB'} + |11\rangle_{BB'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{CC'} + |11\rangle_{CC'})$$

What happens if we do a local measurement (in *your lab*) in the GHZ basis of the three particles at A, B, C ?

Hint : The states of the 8 dimensional basis of entangled GHZ states are $\frac{1}{\sqrt{2}}(|000\rangle_{ABC}+|111\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|000\rangle_{ABC}-|111\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|001\rangle_{ABC}+|110\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|001\rangle_{ABC}-|110\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|010\rangle_{ABC}+|101\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|010\rangle_{ABC}-|101\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|100\rangle_{ABC} + |011\rangle_{ABC})$, $\frac{1}{\sqrt{2}}(|100\rangle_{ABC} - |011\rangle_{ABC})$.