Homework 12, December 2013. Quantum information theory and computation

Problem 1. Quantum Toffoli gate

The quantum Toffoli gate is a double-control-not gate denoted by CC-NOT. On computational basis states it acts as in the classical case. There are three input and three output bits. The third input bit is flipped iff the other two (control bits) are both equal to 1. Check that the quantum Toffoli gate can be obtained from the circuit in figure 4.9 of Chuang and Nielsen page 182, chap 4. This circuit involves the set $\{T, S, H, CNOT\}$. Definitions of these gates is on page 177 and 178 of same chap.

Problem 2. Deutsch-Josza problem

Check the calculations of the course leading to the formula for the probability of outcome equal to $(a_1, ..., a_N) = (0, ..., 0)$,

$$Prob(0, ..., 0) = \frac{1}{2^{2N}} \Big| \sum_{b_1,...,b_N} (-1)^{f(b_1,...,b_N)} \Big|^2$$

Problem 3. Unitarity of QFT

Prove that the Quantum Fourier Transform

$$|x\rangle \to \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |y\rangle$$

is unitary.

Problem 4. Tensor product decomposition of QFT

The binary representation of $x \in \{0, ..., N-1\}$ for $N = 2^n$ is

$$x = x_{n-1}.2^{n-1} + ... + x_2.2^2 + x_1.2^1 + x_0.2^0, \qquad x_i \in \{0, 1\}$$

Prove that the QFT of $|x\rangle$ is equal to the tensor product

$$(|0\rangle + e^{\pi i \frac{x}{2^0}} |1\rangle) \otimes (|0\rangle + e^{\pi i \frac{x}{2^1}} |1\rangle) \otimes ... \otimes (|0\rangle + e^{\pi i \frac{x}{2^{n-1}}} |1\rangle)$$

Hint: In the QFT of $|x\rangle$ represent $|y\rangle = |y', y_0\rangle$ where $y_0 = 0$ or $y_0 = 1$ and inspect each contribution $y_0 = 0$ or 1.