ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
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Handout 3 Statistical Physics for Communication and Computer Science
Homework 3 March 6, 2013

The goal of this homework is to discuss the statistical mechanical formulation of the random K-SAT problem. We consider the ensemble of random formulas $\mathcal{F}(n, K, M)$ defined in chapter one (in class). The clause density will be denoted $\alpha = M/n$. In the first problem you will write the Hamiltonian and the statistical mechanical measures in the spin language. In the second problem you will derive a very elementary upper bound on the sat-unsat phase transition threshold α_s. Hint: there are no big calculations in this homework.

Given a formula $F \in \mathcal{F}(n, K, M)$ consider the following cost function:

$$\mathcal{H}_F(x_1, ..., x_n) = \text{number of clauses violated by the assignment } x_1, ..., x_n. \tag{1}$$

This is our Hamiltonian or energy function (x_i the Boolean variables).

Problem 1 (Hamiltonian, microcanonical measure, finite temperature Gibbs measure). Introduce the "spin" variables $s_i = (-1)^{x_i}$ that take values in $\{-1, +1\}$. Furthermore if clause c_a contains x_i associate $J_{ai} = +1$, and if it contains $\bar{x}_i$ associate $J_{ai} = -1$. Thus full edges have $J_{ai} = +1$ and dashed edges have $J_{ai} = +1$, and J_{ai} are Bernoulli(1/2).

(a) Verify that each clause contributes a term

$$\prod_{i \in c_a} \left(\frac{1 + s_i J_{ia}}{2}\right) \tag{2}$$

and then, write down the Hamiltonian or energy function in the spin language.

(b) Explain in one sentence which are dynamical variables and which are the frozen (or equivalently quenched) random variables in the problem.

(c) Show that the following counts the number of solutions of F

$$Z = \sum_{s_1,...,s_n \in \{-1,+1\}^n} \prod_{a=1}^{M} \left(1 - \prod_{i \in c_a} \left(\frac{1 + s_i J_{ia}}{2}\right)\right) \tag{3}$$

(d) Convince yourself that the microcanonical measure for the zero-energy surface is nothing else than the uniform measure over solutions of F. Also, convince yourself that Z is the partition function (normalization factor) of the microcanonical zero-energy measure. Note that this measure is well defined only if F admits at least one solution.

(d) Now take the Hamiltonian found in question (a) and write down the Gibbs measure for inverse temperature β. Note that this measure has the advantage that it is always well defined, i.e even if F does not have a solution. Consider the free energy $f_F(\beta)$ (normalized by the number of variables) for a fixed formula F. Show that

$$\lim_{\beta \to +\infty} \beta^{-1} f_F(\beta) = \frac{1}{n} \min_{\underline{x}} \mathcal{H}(\underline{x}) \tag{4}$$

This formula is interesting because if we succeed in computing the free energy and if its zero temperature limit is non zero, then we can deduce that F is unsat. The catch is that computing the free energy is a difficult problem.

Problem 2 (Crude upper bound on α_s). Below $\mathbb{P}$ and $\mathbb{E}$ are with respect to the random ensemble $\mathcal{F}(n, K, M)$. Consider the partition function Z of the microcanonical ensemble.

a) Show the Markov inequality $\mathbb{P}[F \text{ satisfiable}] \leq \mathbb{E}[Z]$.

b) Show that

$$\mathbb{E}[Z] = 2^n(1 - 2^{-K})^M. \tag{5}$$

c) Deduce the upper bound

$$\alpha_s < \frac{\ln 2}{|\ln(1 - 2^{-K})|}. \tag{6}$$

For $K = 3$ this yields $\alpha_s(3) < 5.191$. It is conjectured that $\alpha_s(3) \approx 4.26$: this value is the prediction of the highly sophisticated cavity method of spin glass theory. The asymptotic behavior of this simple upper bound for $K \to +\infty$ is $2^K \ln 2$, which is known to be tight. However, the large K corrections obtained by this bound are not tight.