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# Lecture Notes 3: Ising Model

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In this lecture we review the Ising model which provides a fundamental paradigm of *discrete statistical mechanics*. Some of the ideas introduced in this chapter will be encountered again later.

## 1  (Pairwise) Ising Model on a General Graph

Let $G = (V, E)$ be an undirected graph with set of vertices $V$ and set of edges $E$. We denote a vertex by $i \in V$ and an edge by $\langle i, j \rangle \in E$. We assign to each vertex $i$, a binary variable $s_i \in \{\pm 1\}$ (we also denote the spin direction by $\uparrow$ and $\downarrow$). We call $(s_1, \ldots, s_{|V|}) = \underline{s}$ the *spin configuration*. The space of all configurations is $\{-1, 1\}^{|V|}$. The Hamiltonian (or energy function, or cost function) of the model is

$$\mathcal{H}(\underline{s}) = -J \sum_{\langle i,j \rangle \in E} s_i s_j - H \sum_{i \in V} s_i, \tag{1}$$

where $J$ (in coding and K-SAT related to check constraints) is the pairwise interaction constant and $H$ is an experimental parameter (external magnetic field acting on the system) that can be tuned (in coding it is random and its distribution depends on the channel noise intensity). For $J > 0$: the interaction is called ferromagnetic. For $J < 0$: the interaction is called antiferromagnetic.

**Side remarks**

- A more general model

$$\mathcal{H}(\underline{s}) = - \sum_{\langle i,j \rangle \in E} J_{ij} s_i s_j - \sum_{i \in V} H_i s_i, \tag{2}$$

  where $J_{ij}$'s and $H_i$'s are *random variables* is called *spin glass* model.

- The values of $\underline{s}$ at which the energy is minimized are called *ground state* or *minimal energy configurations*. In the case where $h = 0$ we have

  $J > 0$ : $i\overset{\uparrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\uparrow}{\bullet}j$ and $i\overset{\downarrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\downarrow}{\bullet}j$ are energetically favored with respect to $i\overset{\uparrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\downarrow}{\bullet}j$ and $i\overset{\downarrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\uparrow}{\bullet}j$.
  $\min_{\underline{s}} \mathcal{H}(\underline{s})$ is attained for $\underline{S} =$ all $+1$ and $\underline{s} =$ all $-1$.

  $J < 0$ : is much more difficult. On an edge $i\overset{\uparrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\downarrow}{\bullet}j$ and $i\overset{\downarrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\uparrow}{\bullet}j$ are energetically favored with respect to $i\overset{\uparrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\uparrow}{\bullet}j$ and $i\overset{\downarrow}{\bullet}\!\!\rule[0.3em]{2em}{0.4pt}\!\!\overset{\downarrow}{\bullet}j$. For a square grid minimal energy configurations are given by checkerboard configurations. But in more general cases for example, if the graph contains triangles, the energies of all three edges cannot be simultaneously minimized.
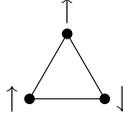
This is called *frustration* which leads to difficult optimization problems.

- Frustration also occurs with random $J_{ij} = \pm J$, which is the case for spin glass model.

## Gibbs Measure

**Definition 1.** *The Gibbs measure or distribution (indexed by the size of graph) is defined*

$$\mu_V(\underline{S}) = \frac{e^{-\beta \mathcal{H}(\underline{s})}}{Z_V}. \tag{3}$$

*The normalization constant $Z_V$ is called the partition function and is defined*

$$Z_V = \sum_{\underline{s} \in \{-1,1\}^{|V|}} e^{-\beta \mathcal{H}(\underline{s})}. \tag{4}$$

The parameter $\beta = \frac{1}{\kappa_B T}$ is proportional to the inverse of the temperature ($k_B$ is the Boltzmann constant). It is common to use the notation $K = \beta J$ and $h = \beta H$.

**Definition 2.** *Free energy per spin*

$$f_V(K, h) = -\frac{1}{|V|} \ln Z_V \tag{5}$$

These two functions depend on $K$ and $h$. The physically correct definition contains an extra $\beta^{-1}$ in front of the ln, but we omit it for convenience.

**Definition 3.** *Average of local functions*

$$\langle s_i \rangle_V(K, h) = \sum_{\underline{S}} s_i \mu_V(\underline{s}) = \sum_{s_i = \pm 1} s_i \mu_{i,V}(s_i) \quad i \in V \tag{6}$$

$$\langle s_i s_j \rangle_V(K, h) = \sum_{\underline{s}} s_i s_j \mu_V(\underline{s}) \quad i, j \in V \times V. \tag{7}$$

## Remarks

Since $s_i$ and $s_i s_j$ take only value $\pm 1$, from $\langle s_i \rangle_V$ and $\langle s_i s_j \rangle_V$ we can reconstruct the full marginals $\mu_{i,V}(s_i)$ and $\mu_{ij,V}(s_i, s_j)$. Indeed we have

$$\begin{cases} \mu_{i,V}(+1) + \mu_{i,V}(-1) = 1, \\ \mu_{i,V}(+1) - \mu_{i,V}(-1) = \langle s_i \rangle_V \end{cases},$$

which allows to reconstruct $\mu_{i,V}(s_i)$. We also have

$$\begin{cases} \mu_{ij}(+1,+1) + \mu_{ij}(+1,-1) + \mu_{ij}(-1,+1) + \mu_{ij}(-1,-1) = 1, \\ \mu_{ij}(+1,+1) - \mu_{ij}(+1,-1) - \mu_{ij}(-1,+1) + \mu_{ij}(-1,-1) = \langle s_i s_j \rangle \\ \mu_{ij}(s_i,+1) + \mu_{ij}(s_i,-1) = \mu_i(s_i) \\ \mu_{ij}(+1,s_j) + \mu_{ij}(-1,s_j) = \mu_j(s_j). \end{cases}$$

Another way to see this is that any function of $s_i$ and $s_i, s_j$ can be written as $f(s_i) = a + bs_i$ and $f(s_i, s_j) = a + bs_i + cs_j + ds_i s_j$ so that $\langle s_i \rangle_V$ and $\langle s_i s_j \rangle_V$ determine all possible average $\langle f(s_i) \rangle_V$ and $\langle f(s_i, s_j) \rangle_V$.

**Definition 4.** *Total magnetization per spin*

$$m_V(K,h) = \frac{1}{|V|} \sum_{i \in V} \langle s_i \rangle_V \tag{8}$$

$$= \frac{\partial}{\partial h} \frac{1}{|V|} \ln Z_V \tag{9}$$

$$= -\frac{\partial}{\partial h} f_V(K,h) \tag{10}$$

We will see that the magnetization is (morally) equivalent to error probability in coding. The reader should check the following very important formulas (take $h_i$ different for each vertex; this defines $\widetilde{Z}_V$)

$$\langle s_i \rangle = -\frac{\partial}{\partial h_i} \ln \widetilde{Z}_V \Big|_{h_i=h}$$

$$\langle s_i s_j \rangle = \frac{\partial^2}{\partial h_i \partial h_j} \ln \widetilde{Z}_V \Big|_{h_i=h}$$

## Large size limit (thermodynamic limit)

We are interested in analyzing the system in the large size limit for a sequence of graphs. This means that we have to specify a sequence of graphs: physically this can be thought as specifying the "shape" of the sample. The large size limit might be shape dependent, boundary condition dependent etc... The simplest case corresponds to taking an Ising model on a cubic grid of equal sizes and let the side go to infinity.

**Theorem 1.** *For a sequence of cubic grids with equal side lengths all the following limits exist*

$$\lim_{|V| \to +\infty} \frac{1}{|V|} \ln Z_V = f(K,h)$$

$$\lim_{|V| \to +\infty} \langle s_i \rangle(K,h)_V = \langle s_i \rangle(K,h)$$

$$\lim_{|V| \to +\infty} \langle s_i s_j \rangle_V = \langle s_i s_j \rangle(K,h)$$

$$\lim_{|V| \to +\infty} \frac{1}{|V|} \sum_{i \in V} \langle s_i \rangle_V = m(K,h)$$

If the side lengths of a rectangular grid do not tend to infinity at the same speed it is still true that the limits exist. However their value may be different.

## 2   Notion of Phase Transition

For a finite graph $G = (V, E)$, $Z_V$ is the sum of finite number of positive terms hence analytic (infinitely differentiable) function in $K, h \in \mathbb{R}$. Similarly, $\ln Z_V$, $\langle s_i \rangle_V$, and $\langle s_i s_j \rangle_V$ are analytic in $K, h \in \mathbb{R}$. Therefore, there is no phase transition for a finite system. However, phase transitions appear in the thermodynamic limit in which $\frac{1}{|V|} \ln Z_V$, $\langle s_i \rangle_V$, and $\langle s_i s_j \rangle_V$ develop singularities as the system size tends to infinity, $|V| \to +\infty$.

**Theorem 2.**

    *a)* $\frac{1}{|V|} \ln Z_V$ *is a convex and analytic function of* $K, h \in \mathbb{R}$.

    *b)* *If* $\lim_{|V| \to \infty} \frac{1}{|V|} \ln Z_V \equiv f(K, h)$ *exists then it is a convex and continuous function of* $K, h \in \mathbb{R}$.

*Proof.*

    a) Analyticity is trivial as explained above. Convexity (separately in $K$ and $h$) follows from

$$\frac{\partial^2}{\partial h^2} \left( \frac{1}{|V|} \ln Z_V \right) = \langle (\sum_i s_i)^2 \rangle - \langle \sum_i s_i \rangle^2 \geq 0$$

$$\frac{\partial^2}{\partial K^2} \left( \frac{1}{|V|} \ln Z_V \right) = \langle ( \sum_{\langle i,j \rangle \in E} s_i s_j)^2 \rangle - \langle \sum_{\langle i,j \rangle \in E} s_i s_j \rangle^2 \geq 0.$$

    b) If the limit of a convex sequence on $\mathbb{R}$ exists, it is convex and continuous.

<div align="right">□</div>

    **Remark:** Free energy is concave because of the minus sign. This is an important condition. If an approximate theory yields a non-concave function something has to be done to "correct" the approximate theory. The most naive (and successful) way of "correcting" is to take the concave hull of the approximation. This was already understood by Maxwell and is called a "Maxwell construction". Note that the concave hull is still an approximation (usually) but its a better one.

### Ehrenfest classification of phase transition

Phase transitions can be classified according to derivatives of $f(K, h)$ (we warn the reader that this is just one possible classification. It is the most usual one, but there exist other more "modern" ones).

**First order:** the first derivative of $f(K, h)$ is discontinuous. We also have

$$m(K, h) \equiv \lim_{|V| \to +\infty} \frac{1}{|V|} \sum_{i \in V} \langle s_i \rangle_V = -\frac{\partial}{\partial h} f(K, h)$$

so the total magnetization per spin, is discontinuous. Figure 1a shows a phase transition of first order.



(a) First order          (b) Second order
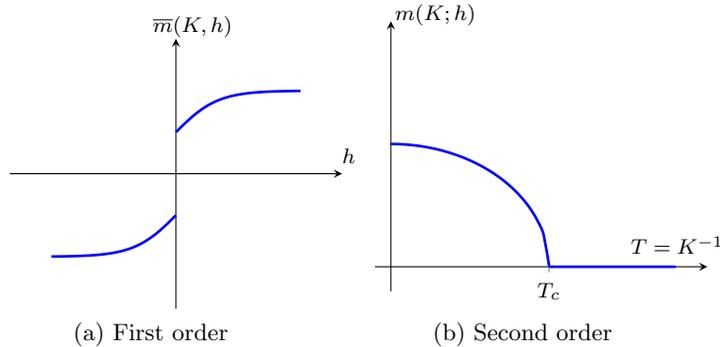
Figure 1: Ehrenfest classification of phase transition

**Second order:** the second derivative of $f(K, h)$ is discontinuous. Then $m(K, h)$ is continuous but its first derivative is discontinuous. This typically happens when the temperature is varied. Figure 1b shows a phase transition of second order.

Evidently one can define higher order transitions within this classification scheme. Even infinite order phase transitions exist where the free energy is infinitely differentiable but not analytic. This hardly manifests itself on the function, but affects correlation functions. We also point out that there exist "more modern" classification schemes in terms of the symmetry changes that occur at a transition.

## 3 Model on a Complete Graph (Curie-Weiss Model)

For certain graphs the Ising model can be exactly solved and analyzed. Two examples are the one dimensional Ising model and the model on trees which we solve in homework. Another case is the complete graph. The model on a complete graph is also called Curie-Weiss model. Let $G = (E, V)$ be a complete graph on $N$ vertices. A complete graph where $N = 4$ is shown in Figure2.
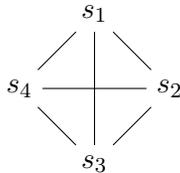


Figure 2: A complete graph with 4 nodes

The Hamiltonian of the system is

$$\mathcal{H}(\underline{s}) = -\frac{J}{N} \sum_{\langle i,j \rangle \in E} s_i s_j - H \sum_{i \in V} s_i. \tag{11}$$

Note that there are $\frac{N(N-1)}{2}$ edges. The interaction constant is scaled by $N$ which is necessary to have well defined large size limit. One way of seeing this is to count the number of terms in both sums in Hamiltonian.

- Interactions $\approx \frac{1}{N}(\frac{N(N-1)}{2}$ terms $) \approx O(N)$.

- $H \sum_i s_i \approx O(N)$.

So the typical value of both sums is $O(N)$. Another way to justify the scaling is to notice that the interactions are not local (local means each spin interacts with an $O(1)$ number of other spins). This has to canceled off to get decent thermodynamic behavior. We are going to calculate the free energy, magnetization and analyze phase transitions for this model.

## Computation of the free energy

There are many ways to accomplish this. We have

$$Z_N = \sum_{\underline{s} \in \{-1,1\}^N} e^{\frac{K}{N} \sum_{\langle i,j \rangle \in E} s_i s_j + h \sum_{i \in V} s_i} \tag{12}$$

Let $m = \frac{1}{N} \sum_{i \in V} s_i$ which is the total magnetization of the spin configuration. Because the graph is complete we have,

$$\sum_{\langle i,j \rangle \in E} s_i s_j = \overbrace{\frac{1}{2}}^{\text{cancels double products}} (\sum_{i \in V} s_i)^2 - \underbrace{\frac{1}{2}N}_{\text{cancels diagonal terms}} = \frac{N^2}{2}m^2 - \frac{N}{2}$$

Rewriting the Hamiltonian, we get

$$\beta \mathcal{H}(\underline{s}) = -N(\frac{K}{2}m^2 + hm) + \frac{K}{2}, \tag{13}$$

and

$$Z_N = e^{\frac{K}{2}} \sum_{\underline{s} \in \{-1,1\}^N} e^{N(\frac{K}{2}m^2 + hm)}. \tag{14}$$

Let $N_+$ and $N_-$ be the number of positive and negative spins respectively. We have

$$\begin{cases} N_+ + N_- = N \\ N_+ - N_- = mN \end{cases} \Rightarrow N+ = (\frac{1+m}{2})N$$

Now in Equation 14 we can carry out the sum over $m = \{\frac{m}{N}; -N, \ldots, +N\}$ and rewrite

$$Z_N = e^{\frac{K}{2}} \sum_m \underbrace{\#(\underline{s} : \sum_i s_i = mN)}_{\binom{N}{N_+} = \binom{N}{N(\frac{1+m}{2})}} e^{N(\frac{K}{2}m^2 + hm)}.$$

using Stirling's formula (exercise), we can approximate $\binom{N}{N(\frac{1+m}{2})} \approx e^{Nh_2(\frac{1+m}{2})}$ when $N \to \infty$, where $h_2$ is the binary entropy function given by $h_2(p) = -p \ln p - (1-p) \ln(1-p)$. This leads to

$$Z_N \sim e^{\frac{K}{2}} \sum_m e^{N(\frac{K}{2}m^2 + hm + h_2(\frac{1+m}{2}))}. \tag{15}$$

This is a Riemann sum. Write $1 = N(\frac{m+1}{N} - \frac{m}{N}) = N dm$ and let $N \to \infty$. One gets

$$Z_N \sim e^{\frac{K}{2}} N \int_{-1}^{+1} e^{N(\frac{K}{2}m^2 + hm + h_2(\frac{1+m}{2}))} dm \tag{16}$$

(All this can be proved rigorously if wished). The free energy is

$$f(K, h) = -\lim_{N \to +\infty} \frac{1}{N} \ln Z_N$$

$$= -\lim_{N \to +\infty} \frac{1}{N} \ln \int_{-1}^{+1} e^{N(\frac{K}{2}m^2 + hm + h_2(\frac{1+m}{2}))} dm$$

$$\overset{\text{Laplace}}{=} -\max_m \left\{ \frac{K}{2}m^2 + hm + h_2(\frac{1+m}{2}) \right\}$$

$$= \min_m \left\{ -(\frac{K}{2}m^2 + hm) - h_2(\frac{1+m}{2}) \right\}$$

$$\boxed{\textbf{Free energy: } f(K, h) = \min_m \left\{ \underbrace{-(\frac{K}{2}m^2 + hm)}_{u(m): \text{ energy}} - \underbrace{h_2(\frac{1+m}{2})}_{s(m): \text{ entropy}} \right\}} \tag{17}$$

**Remarks:**

- Models on complete graphs, on trees, on random graphs and more generally models on "infinite dimensional graphs" lead to a variational form for the free energy. The reason is that we can reduce the number of degrees of freedom to some small number of global object (like $m$ in Equation 17) and the statistical sum becomes a small number of integrals. The "functional" $u(m) - s(m)$ (such as in Equation 17) is called a *Landau free energy* or also (in the context of coding and computer science) a *Bethe free energy*. This is always of the form of the difference between an *internal* energy $u(m)$ and an *entropy* $s(m)$, and is similar to the thermodynamic relation between free energy, internal energy and entropy (see also lecture 2, paragraph on equivalence of ensembles). We will see more examples of Bethe free energies in the following lectures.

- For models on finite dimensional graphs (e.g. a cubic grid in three dimensions) this reduction in number of degrees of freedom does not occur and one is confronted to the full statistical sum. The solution will not be given by a variational problem: in general it is not known how to find it, except when special symmetries are present. Somewhat hidden symmetries are present in various two dimensional models which are exactly solvable. The Ising model on the square grid is one of them. However in three dimensions the problem of finding exact solutions is essentially wide open.

- Note that $f(K, h)$ given by the variational problem above is concave in $K$ and $h$ (say why ? Exercise.).

## Curie-Weiss fixed point equation

The variational problem can be solved by looking at critical points $\frac{d}{dm}(u(m) - s(m)) = 0$:

$$Km + h\frac{d}{dm}h_2(\frac{1+m}{2}) = 0$$
$$Km + h + \frac{1}{2}\ln\frac{1-m}{1+m} = 0$$
$$Km + h = \operatorname{atanh} m$$

The last equation is obtained from the identity $\tanh(\frac{1}{2}\ln\frac{1+m}{1-m}) = m$. The Curie-Weiss fixed point equation also called *mean field equation* is

$$\boxed{m = \tanh(Km + h)} \tag{18}$$

Of course this equation may have many solutions. The correct one is the one which minimizes the Landau/Bethe free energy $u(m) - s(m)$. We call it $\overline{m}(K, h)$.

### Remark:

Often one reads that this is a *mean field equation*. This is because it is obtained by a *mean field approximation* method on $\mathbb{Z}^d$. The $Km$ is the "mean field" felt by a spin due to the magnetization of its neighbors. This will be clarified in later chapters of the course.

### Interpretation of $\overline{m}(K, h)$

$$f(K, h) = \min_m G(m; h) = G(\overline{m}(K, h); h)$$

where

$$G(m; h) = u(m) - s(m)$$

$$= -(\frac{k}{2}m^2 + hm) - h_2(\frac{1+m}{2}).$$

By Calculating the derivative

$$-\frac{\partial f}{\partial h} = -\frac{d}{dh}G(\overline{m}(K,h);h)$$

$$= -\underbrace{\frac{\partial G}{\partial m}\bigg|_{\overline{m}}}_{0\text{ at }\overline{m}} \underbrace{\frac{d\overline{m}}{dh}}_{\text{bounded}} - \underbrace{\frac{\partial G}{\partial h}\bigg|_{\overline{m}}}_{\overline{m}(K,h)}$$

$$= \overline{m}(K,h),$$

we get

$$\boxed{\overline{m}(K,h) = -\frac{df}{dh} = \text{ total magnetization per spin.}} \tag{19}$$

This can also be shown by directly calculating $\langle s_i \rangle_V$ by the previous method (reducing the statistical sum to one integral over $m$.

## Analysis of the Curie-Weiss equation and phase transitions

We can visually determine the solutions by looking at intersection of both sides of the equation $m = \tanh(Km + h)$.
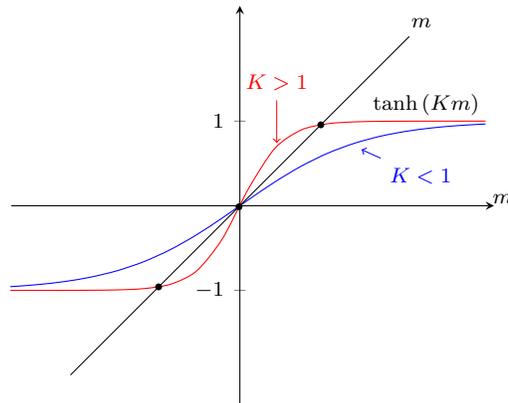


Figure 3: Curie-Weiss fixed points, $h = 0$

**Case $h = 0$**

$K < 1$ : unique solution, $\overline{m}(K,0) = 0$, $\quad f(K,0) = \ln 2$. (true free energy $f(K,0) = k_B T \ln 2$)

$K > 1$ : three solutions $\{\overline{m}_-, 0, \overline{m}_+\}$, two extremes are minimizer of $u(m) - s(m)$

The fixed points are shown in Figure 3 and the corresponding free energy is shown in Figure 4.

(a) $K > 1$        (b) $K < 1$
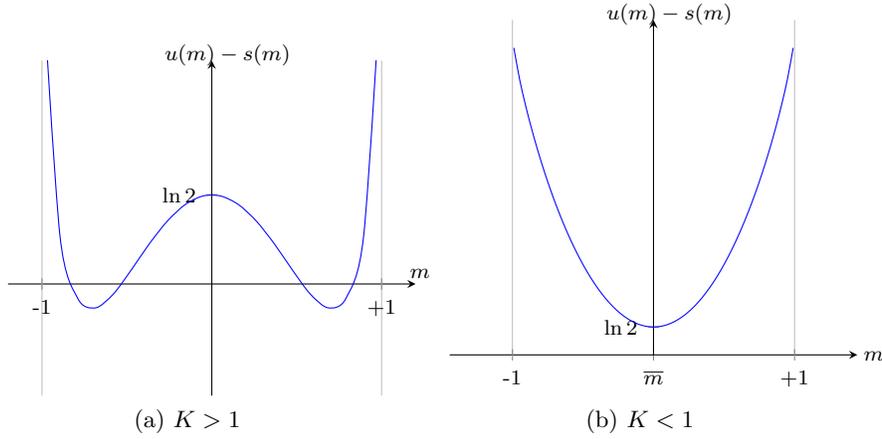
Figure 4: Landau/Bethe free energy $u(m) - s(m)$

**Phase transition as a function of $K$:** Second order phase transition at $K_c = 1$. It is shown in Figure 5. $K^{-1}$ is proportional to temperature $T$.
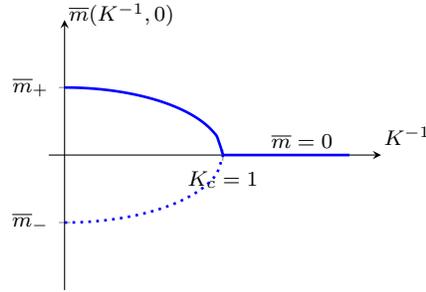


Figure 5: Phase transition as a function of $T$ in Curie-Weiss model

**Critical behavior:** for $K$ close to $K_c = 1$ we have $\overline{m}$ small, so we can expand the Curie-Weiss equation

$$m = tanh(Km) \approx Km - \frac{K^3}{3}m^3,$$

besides the trivial solution $\overline{m} = 0$ we have

$$1 \approx K - \frac{K^3}{3}m^2$$

$$\Rightarrow m^2 \approx \frac{3}{K^3}(K - 1)$$

$$\Rightarrow m \sim \pm 3(K - 1)^{\frac{1}{2}}$$

$$\Rightarrow m \sim \pm 3(K - K_c)^{\frac{1}{2}}$$

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The exponent $\frac{1}{2}$ is called *critical exponent*. Remarkably it often does not depend on the detailed form of the Hamiltonian but only on such things as the dimensionality of the system (here $d = +\infty$), and the underlying symmetries of the Hamiltonian (here $s_i \to \pm s_i$).

**Case** $h > 0$  Fixed points and free energy are shown in Figure 6 where $h > 0$ ($h$ not too large) and $K > 1$.



(a) Fixed points
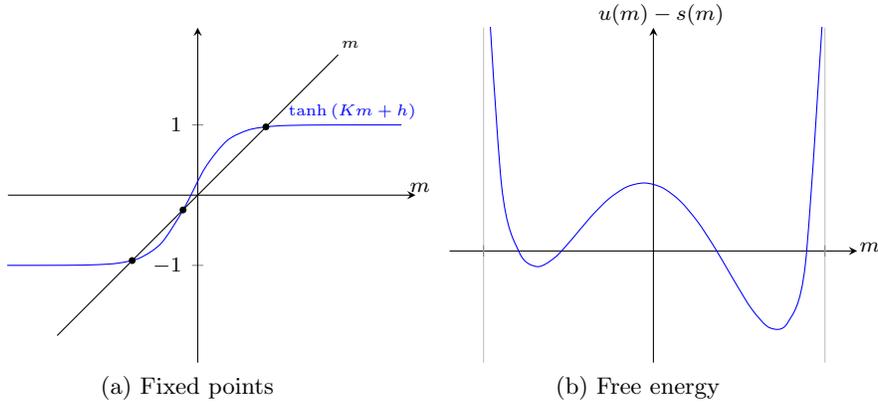
(b) Free energy

Figure 6: Curie-Weiss fixed points, $h > 0, K > 1$

Fixed points and free energy are shown in Figure 7 where $h > 0$ ($h$ large) and $K < 1$.



(a) Fixed points
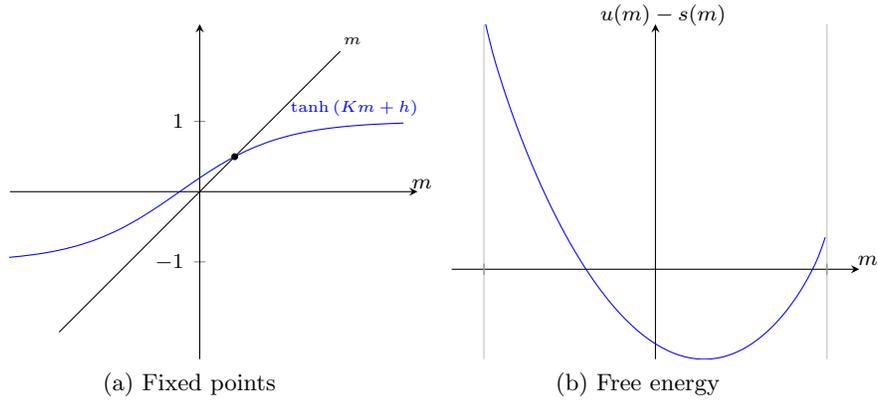
(b) Free energy

Figure 7: Curie-Weiss fixed points, $h > 0, K < 1$

Note that the global minimizer $\overline{m} > 0$.

**Case** $h < 0$  Fixed points and free energy are shown in Figure 8 where $h < 0$ ($h$ not too large) and $K > 1$.

(a) Fixed points  (b) Free energy
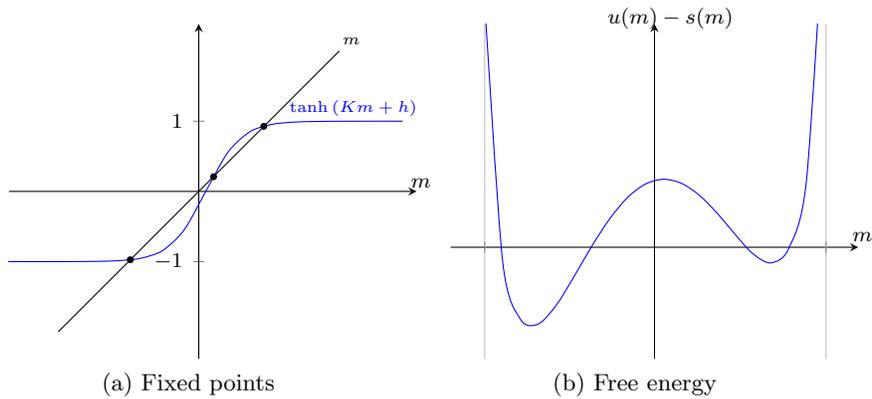
Figure 8: Curie-Weiss fixed points, $h < 0, K > 1$

Fixed points and free energy are shown in Figure 9 where $h < 0$ ($h$ large) and $K < 1$.



(a) Fixed points  (b) Free energy
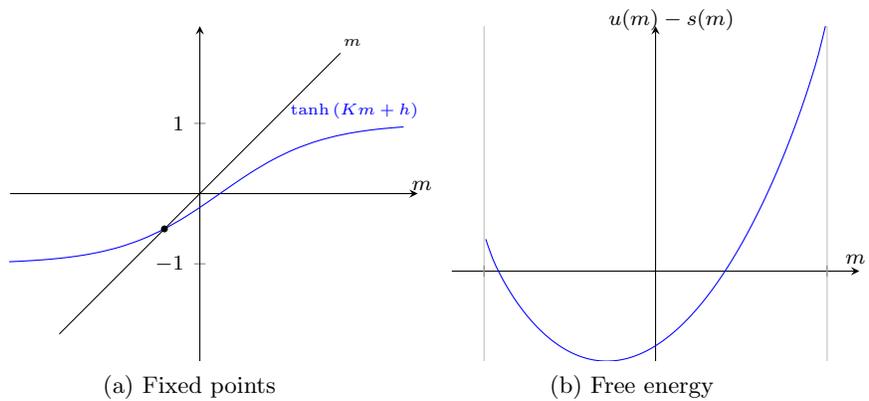
Figure 9: Curie-Weiss fixed points, $h < 0, K < 1$

Note that the global minimizer $\overline{m} < 0$.

**Phase Transition as a function of $h$:** Summarizing, we see that for $K > 1$ at $h = 0$, $\overline{m}(K, h)$ is discontinuous. This is a transition of first order. Now, at $K = K_c = 1$ the jump disappears and the curve behaves as $|h|^{\frac{1}{3}}$ as $h \to 0$: this is again an example of a second order phase transition with critical exponent $\frac{1}{3}$ (exercise: show this by expanding the Curie-Weiss equation for small $h$ when $K = K_c = 1$.)
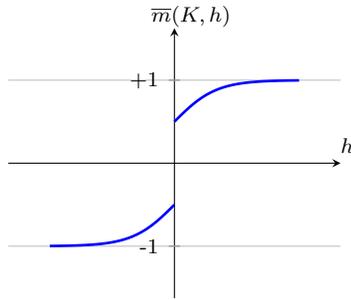
Figure 10: Phase transition in Curie-Weiss model when $K > 1$ as a function of $h$. The phase transition is at $h = 0$

**The phase diagram:** The following picture summarizes the nature of the phase transitions in the $(K^{-1}, h)$ or $(T, h)$ plane.
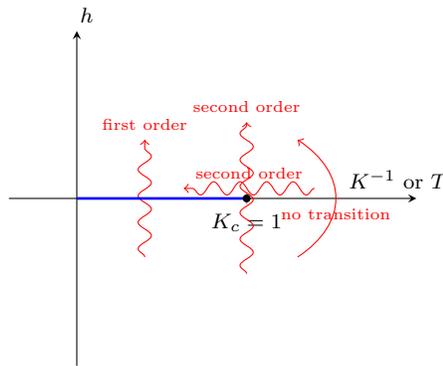


Figure 11: The line is called *coexistence line* because two thermodynamic phases (e.g. water/ice) coexist for parameters on it. Crossing the thick line is a first order phase transition. This line is terminated by the *critical point*. Crossing the critical point is a second order phase transition. There are many ways to cross it.

# 4 General picture for the model on $\mathbb{Z}^d$

We will essentially not be concerned with the remarks in this section, but we give them for completeness. For the standard Ising model on the grid $\mathbb{Z}^d$ here is what is known. After one century of research the analysis of this model still offers deep mathematical problems.

## Dimensionality dependence

- $d = 1$: No phase transitions (except for interactions with very large range) (Ising 1920).

- $d \geq 2$: First and second order phase transition are present. Qualitatively these are much like those of Curie-Weiss model (Proofs of existence of transition by Peierls 1935,

Griffith, Dobrushin 1965-70). Note however that the critical exponents of second order phase transitions are not the same than in Curie-Weiss.

- $d \to +\infty$: same solution than on the complete graph.

## Critical behavior

- $d \geq 4$: exponents of second order transition $\frac{1}{2}$ and $\frac{1}{3}$, as the ones found before. Remarkably they do not depend on the microscopic structure of $\mathcal{H}(\underline{s}.)$

- $d = 2, 3$ other critical exponents for the second order transition. For example, for $d = 2$, $m \sim |K - K_C|^{\frac{1}{8}}$. This results from Onsager's famous exact solution of the two dimensional model (1944). For $d = 3$, computing those is the subject of the *renormalization group* which was developed in the 70's (Wilson, Fisher, Kadanoff. Nobel prize to K. Wilson). Their exact values are unknown however as one has to use expansions and numerical calculations.

## Spontaneous magnetization on $\mathbb{Z}^d$

If one sets $h = 0$ from the outset, one has $\langle s_0 \rangle_V = 0$ and this is also true for the limit $|V| \to +\infty$. However for $K > 1$ the limits $h \to 0$ and $|V| \to +\infty$ do not commute. One defines the spontaneous magnetization as

$$m_\pm = lim_{h \to 0_\pm} \lim_{|V| \to +\infty} \frac{1}{|V|} \sum_{i \in V} \langle s_0 \rangle$$

$$= lim_{h \to 0_\pm} \lim_{|V| \to +\infty} \langle s_0 \rangle_V$$

On the coexistence line (see phase diagram) it is discontinuous. This means that for $K > 1$ an infinitesimally positive magnetic field tilts typical spin configurations to mostly $+1$'s and an infinitesimally negative magnetic field tilts typical spin configurations to mostly $-1$'s. In nature a magnet (say) picks up one of the two limits because of infinitesimal magnetic fields that perturb the sample: this phenomenon is called *spontaneous symmetry breaking.*

## Infinite volume Gibbs measures

One can study higher marginals/higher moments in the infinite size limit. From those one may reconstruct an infinite volume Gibbs measure. Just as the spontaneous magnetization defined just above does not have a unique limit on the phase transition line , the limiting Gibbs measure is also non-unique. Thus characterizing the set of infinite volume Gibbs measures is a highly non-trivial problem. This is a convex set. Extremal measures are called *pure states* and describe pure thermodynamic phases (all water/all ice for example). Convex combination of extremal measures describe the coexistence of pure thermodynamic phases (the coexistence of water and ice for example).

- In $d = 1$ only one infinite volume Gibbs measure for any finite temperature: no phase transition. The convex set is a point.

- In $d = 2$ only two extremal measures for $K > K_c$ and $h = 0$ (proved in the 80's). The convex set is a segment. For other points of the phase diagram there is only one measure (convex set is a point). At the critical point $K = K_c$ and $h = 0$ the problem is different: seen from large scales the typical configurations look fractal and self-similar. This is the subject of conformal invariance developed by physicist in the 80's. Predictions of conformal invariance have been recently proved by mathematicians (Fields medals in 2006 to Werner, Okhounov and 2010 to Smirnov).

- $d = 3$ On the first order line it is known that there exist more than two extremal states. The convex set is more rich than in two dimensions. There are extremal Gibbs measures that describe states with interfaces. Interfaces are stable in three dimensions (and not in $d \leq 2$).