Lecture 13: Energetic Cavity method for K-SAT. Survey Propagation inspired decimation.

1. Zero-Temperature Cavity Method.

We apply the cavity method developped in lecture 12 to K-SAT (for the SAT phase only). Given an instance F of a formula we want to study the uniform measure over SAT solutions

$$\mu_F(\vec{x}) = \frac{\mathbb{1}(\vec{x} \text{ SAT's } F)}{Z_F}.$$

However to use the concepts discussed so far (and because this measure is ill-defined for assignments that do not satisfy F we consider first a finite temperature version of the problem:

$$\mu_{\beta,F}(\vec{x}) = \frac{1}{Z} \prod_a e^{-\beta E_a(x_{\partial a})}$$

where $e^{-\beta E_a(x_{\partial a})} = e^{-\beta(1 - f_a(x_{\partial a}))}$ and

$$Z_{\beta,F} = \sum_{\vec{x}} \prod_a e^{-\beta E_a(x_{\partial a})}.$$

One sees that as $\beta \to +\infty$, $\mu_\beta(\vec{x}) \to \mu_F(\vec{x})$.

Indeed if $f_a(\vec{x}) = 0$, $\exp(-\beta(1 - f_a(\vec{x}))) \to 0$ and for $f_a(\vec{x}) = 1$, $\exp(-\beta(1 - f_a(\vec{x}))) = 1$. Thus

$$e^{-\beta E_a(\vec{x})} \longrightarrow f_a(\vec{x}) \quad \text{as } \beta \to +\infty.$$

We apply the cavity formalism in the limit $\beta \to +\infty$ (assuming that it still holds in this limit).

Min-sum fixed point equations.

When $\beta \to +\infty$ the BP fixed point equations become min-sum fixed point equations. Let us show it explicitly. Recall

$$\nu_{i\to a}(x_i) \propto \prod_{b\in\partial i\setminus a} \hat{\nu}_{b\to i}(x_i)$$

$$\hat{\nu}_{a\to i}(x_i) \propto \sum_{\sim x_i} f_a(x_{\partial a}) \prod_{j\in\partial a\setminus i} \nu_{j\to a}(x_j).$$

We parametrise messages as follows:

$$\nu_{i\to a}(s_i) = \frac{\exp(\beta h_{i\to a} s_i)}{2\cosh \beta h_{i\to a}} \overset{\beta\to+\infty}{\approx} \exp \beta\underbrace{(h_{i\to a} s_i - |h_{i\to a}|)}_{\equiv -E_{i\to a}(s_i)} \equiv \exp\left(-\beta E_{i\to a}(s_i)\right)$$

and similarly for $\hat{\nu}_{a\to i}(s_i)$.

One gets, taking the log of BP equations and taking $\beta \to +\infty$,

$$\begin{cases} E_{i\to a}(x_i) = \sum_{b\in\partial i\setminus a} \hat{E}_{b\to i}(x_i) - C_{i\to a} \\ \hat{E}_{a\to i}(x_i) = \min_{x_{\partial a\setminus i}} \left\{ E_a(x_{\partial a}) + \sum_{j\in\partial a\setminus i} E_{j\to a}(x_j) \right\} - \hat{C}_{a\to i}. \end{cases}$$

The two constants $C_{i\to a}$ and $\hat{C}_{a\to i}$ come from the log of the normalization factors in the BP equations. It is easy to see that

$$\begin{cases} C_{i\to a} = \min_{x_i}\left\{\sum_{b\in\partial i\setminus a} \hat{E}_{b\to i}(x_i)\right\}, \\ \hat{C}_{a\to i} = \min_{x_i}\min_{x_{\partial a\setminus i}}\left\{E_a(x_{\partial a}) + \sum_{j\in\partial a\setminus i} E_{j\to a}(x_j)\right\} = \min_{x_{\partial a\setminus i}}\left(\sum_{j\in\partial a\setminus i} E_{j\to a}(x_j)\right) \end{cases}$$

Note that we always have $\min_{x_i} E_{i\to a}(x_i) = 0$ & $\min_{x_i} \hat{E}_{a\to i}(x_i) = 0$.

The messages $E_{i\to a}(x_i)$ & $\hat{E}_{a\to i}(x_i)$ will be called "energy costs".

Bethe (zero temperature) energy function.

Recall that the Bethe functional is $F_{\text{Bethe}} = \sum_a F_a + \sum_i F_i - \sum_{ai} F_{ai}$

where

$$F_a = \ln\left\{\sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{i\in\partial a} \nu_{i\to a}(x_i)\right\}$$

$$F_i = \ln\left\{\sum_{x_i} \prod_{b\in\partial i} \hat{\nu}_{b\to i}(x_i)\right\}$$

$$F_{ai} = \ln\left\{\sum_{x_i} \nu_{i\to a}(x_i)\,\hat{\nu}_{a\to i}(x_i)\right\}.$$

Define the Bethe zero-temperature energy function as

$$E_{\text{Bethe}} = \lim_{\beta\to+\infty} \beta^{-1} F_{\text{Bethe}} \equiv \sum_a E_a + \sum_i E_i - \sum_{ai} E_{ai}.$$

We get

$$E_a = \min_{x_{\partial a}}\left\{E_a(x_{\partial a}) + \sum_{i\in\partial a} E_{i\to a}(x_i)\right\}$$

$$E_i = \min_{x_i}\left\{\sum_{b\in\partial i} \hat{E}_{b\to i}(x_i)\right\}$$

and

$$E_{ai} = \min_{x_i} \left\{ E_{i\to a}(x_i) + \hat{E}_{a\to i}(x_i) \right\}.$$

Note that the factors $C_{i\to a}$ and $\hat{C}_{a\to i}$ cancel out.

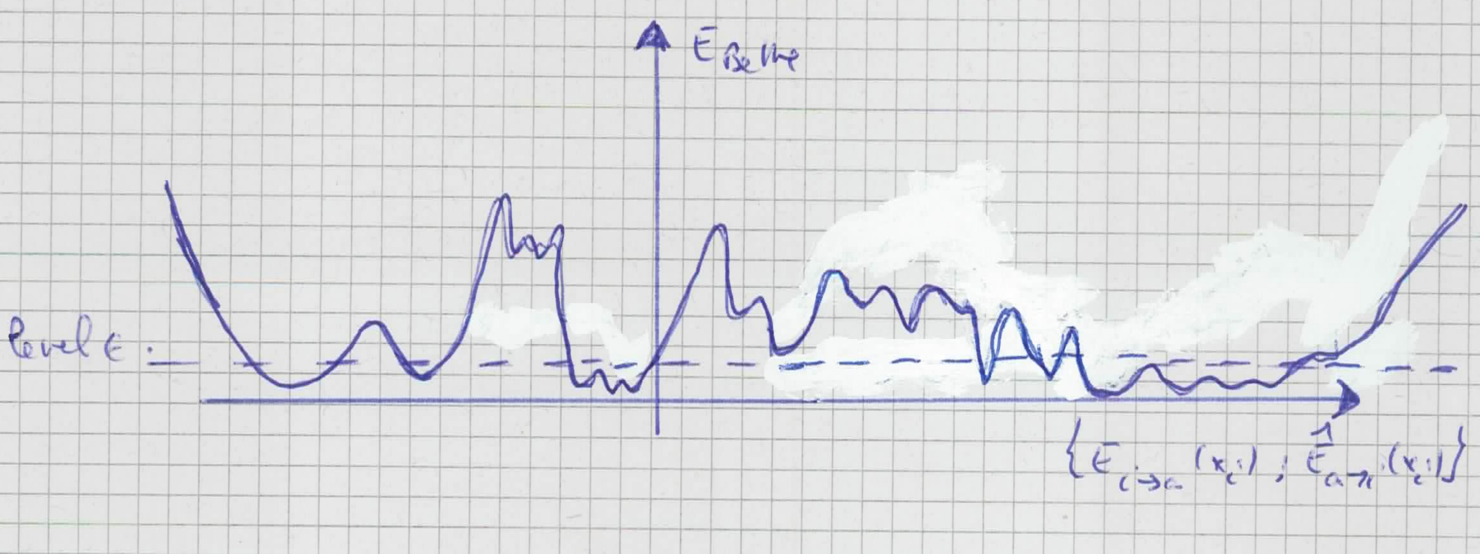
The landscape and Complexity.

We can picture the Bethe zero temperature energy function as a landscape over the space of "energy costs" $\{E_{i\to a}(x_i), \hat{E}_{a\to i}(x_i)\}$. Stationary points satisfy the min-sum equations. We define for fixed ϵ:

$$\Sigma(\epsilon) = \frac{1}{N} \ln \sum_{\substack{E_{i\to a}, \hat{E}_{a\to i} \\ \text{obey} \\ \text{min-sum}}} \delta\left(N\epsilon - E_{Bethe}(E_{i\to a}, \hat{E}_{a\to i})\right).$$

This "complexity" counts points at energy level ϵ.

For constraint satisfaction problems in the SAT phase the minima of the landscape are at $\epsilon \simeq 0$, so that for $\epsilon \simeq 0$ this function counts essentially only minima / pure states / clusters.



This function counts the number of clusters in the phase diagram of K-SAT (for the SAT phase). See picture in page 3 of lecture 12. Note that by going to the limit $\beta \to +\infty$, we have lost the information about the "size" of clusters. Indeed the function $\Sigma(f)$ defined in lecture 12 depends on $\beta^{-1} f = \epsilon - \beta^{-1} s$ (ϵ the energy and s the entropy; recall from lect 3 or 4). The entropy s is the "internal entropy" of a cluster ($\log$ number of solns inside cluster); as $\beta \to +\infty$ we get $f \to \epsilon$.

The zero temperature limit studied here counts clusters irrespective of their size.

This is sometimes called "energetic cavity method". A more refined picture is obtained by taking into account the size of clusters; something that the "entropic cavity method" does.

In order to compute $\Sigma(\epsilon \approx 0)$ we introduce the "Partition Function":

$$\Xi(y) = \sum_{\substack{\{E_{i\to a};\\ \hat{E}_{a\to i}\}\\ \text{minima}}} e^{-y\, E_{\text{Bethe}}[E_{i\to a}, \hat{E}_{a\to i}]}.$$

As $y \to +\infty$, the sum is dominated by the

minima such that $E_{Bethe}[E_{i\to a}, \hat{E}_{a\to i}] = N\epsilon \simeq 0$.

Remark! This is true for the SAT phase of a constraint satisfaction problem. Beyond the SAT phase it is not true.

Thus we can compute the complexity in a SAT phase as follows:

$$\Sigma(\epsilon \simeq 0) = \lim_{y\to+\infty} \ln \Xi(y)$$

The goal of the "energetic cavity method" is to compute the "free energy" $\ln \Xi(y)$ using the Bethe formalism in the limit $y \to +\infty$. We will see that in this limit simplifications occur in the message-passing equations.

Remark also that the natural relation between the parameter x (of Parisi) in lecture II and the parameter y here is

$$x = \beta^{-1} y.$$

Reweighting factors in the limit $\beta \to +\infty$.

We compute pratical expressions for

$$\lim_{\beta \to +\infty} e^{x(F_a - F_{ai})} = \lim_{\beta \to +\infty} e^{y \beta^{-1}(F_a - F_{ai})} = e^{y(E_a - E_{ai})}.$$

and

$$\lim_{\beta \to +\infty} e^{x(F_i - F_{ai})} = \lim_{\beta \to +\infty} e^{y \beta^{-1}(F_i - F_{ai})}.$$

First factor: taking the Bethe expressions we find

$$e^{x(F_a - F_{ai})} = \left\{ \frac{\sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{i \in \partial a} \nu_{i \to a}(x_i)}{\sum_{x_i} \hat{\nu}_{a \to i}(x_i)\, \nu_{i \to a}(x_i)} \right\}^x$$

$$\overset{\text{use BP equ.}}{=} \left\{ \frac{\sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{i \in \partial a} \nu_{i \to a}(x_i)}{\sum_{x_i} \left(\sum_{x_{\partial a \setminus i}} f_a(x_{\partial a}) \prod_{j \in \partial a \setminus i} \nu_{j \to a}(x_j) \right) \cdot \nu_{i \to a}(x_i)} \right\}^x$$

$$\cdot \left\{ \sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{j \in \partial a \setminus i} \nu_{j \to a}(x_j) \right\}^x$$

comes from normalization factor

$$= \left\{ \sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{j \in \partial a \setminus i} \nu_{j \to a}(x_j) \right\}^x.$$

Exponentiating, setting $x = \beta^{-1} y$, and taking $\beta \to +\infty$ we find,

$$\lim_{\beta \to +\infty} e^{\beta^{-1} y (F_a - F_{ai})} = \boxed{\begin{aligned} &\exp\left(-y \min_{x_{\partial a}} \left\{ E_a(x_{\partial a}) + \sum_{j \in \partial a \setminus i} E_{j \to a}(x_j) \right\}\right). \\ &\equiv \exp(-y W_1) . \end{aligned}}$$

Second factor: similarly,

$$e^{x(F_i - F_{ai})} = \left\{ \frac{\sum_{x_i} \prod_{a \in \partial i} \hat{\nu}_{a \to i}(x_i)}{\sum_{x_i} \hat{\nu}_{a \to i}(x_i)\, \nu_{i \to a}(x_i)} \right\}^x$$

using BP equ with correct normalization

$$= \left\{ \frac{\sum_{x_i} \prod_{a \in \partial i} \hat{\nu}_{a \to i}(x_i)}{\sum_{x_i} \hat{\nu}_{a \to i} \cdot \dfrac{\prod_{b \in \partial a \setminus i} \hat{\nu}_{b \to i}(x_i)}{\sum_{x_i} \prod_{b \in \partial a \setminus i} \hat{\nu}_{b \to i}(x_i)}} \right\}^x$$

$$= \left\{ \sum_{x_i} \prod_{b \in \partial a \setminus i} \hat{\nu}_{b \to i}(x_i) \right\}^x$$

Exponentiating, setting $x = \beta^{-1} y$, and taking $\beta \to +\infty$ we find,

$$\lim_{\beta \to +\infty} e^{\beta^{-1} y (F_i - F_{ai})} = \boxed{\begin{aligned} &\exp\left(-y \min_{x_i} \left\{ \sum_{b \in \partial a \setminus i} \hat{E}_{b \to i}(x_i) \right\}\right). \\ &\equiv \exp(-y W_2) . \end{aligned}}$$

2. Summary of SP formalism for zero temperature.

- Applies to one graph instance.

- "Energy costs" $\hat{E}_{a\to i}(x_i)$ & $E_{i\to a}(x_i)$ plays the role of underlying variables.

- "Surveys" $\hat{Q}_{a\to i}(\hat{E}_{a\to i})$ & $Q_{i\to a}(E_{i\to a})$ are equal to the fraction of "pure states" for which the energy costs are $\hat{E}_{a\to i} = (\hat{E}_{a\to i}(0);\hat{E}_{a\to i}(1))$ & $E_{i\to a} = (E_{i\to a}(0); E_{i\to a}(1))$.

- Energy costs satisfy Min-Sum equations:

$$\begin{cases} \hat{E}_{a\to i}(x_i) = \min_{\sim x_i} \left\{ E_a(x_{\partial a}) + \sum_{j\in\partial a\setminus i} E_{j\to a}(x_j) \right\} - \hat{C}_{a\to i} & \text{(MS 1)} \\ E_{i\to a}(x_i) = \sum_{b\in\partial a\setminus i} \hat{E}_{b\to i}(x_i) - C_{i\to a} & \text{(MS 2)} \end{cases}$$

- Surveys satisfy $\overset{\text{BP}}{\vee}$ message-passing with an extra reweighting factors

$$\hat{Q}_{a\to i}(E_{a\to i}) = \sum_{\sim E_{a\to i}} \mathbb{1}(\text{MS 1}) \cdot e^{-y W_1} \prod_{j\in\partial a\setminus i} Q_{j\to a}(E_{j\to a}).$$

$$Q_{i\to a}(E_{i\to a}) = \sum_{\sim E_{i\to a}} \mathbb{1}(\text{MS 2})\, e^{-y W_2} \prod_{b\in\partial i\setminus a} \hat{Q}_{b\to i}(\hat{E}_{b\to i})$$

- The RSB Bethe functional is (calculating $\ln \Xi(y)$)

$$\Phi_{RSB-Bethe}(y) = \sum_a E_a^{RSB} + \sum_i E_i^{RSB} - \sum_{ai} E_{ai}^{RSB} .$$

$$E_a^{RSB} = \ln \left\{ \sum_{\substack{\{E_{i\to a}\} \\ i\in\partial a}} e^{-y E_a} \prod_{i\in\partial a} Q_{i\to a}(E_{i\to a}) \right\} .$$

$$E_i^{RSB} = \ln \left\{ \sum_{\substack{\{\hat{E}_{a\to i}\} \\ a\in\partial i}} e^{-y E_i} \prod_{a\in\partial i} \hat{Q}_{a\to i}(E_{a\to i}) \right\}$$

$$E_{ai}^{RSB} = \ln \left\{ \sum_{\substack{\hat{E}_{a\to i} \\ E_{i\to a}}} e^{-y E_{ai}} Q_{i\to a}(E_{i\to a}) \, \hat{Q}_{a\to i}(E_{a\to i}) \right\} .$$

- The stationary points of this functional satisfy the SP equations (of page 9).

Recall here that:

$$E_a = \min_{x_{\partial a}} \left\{ E_a(x_{\partial a}) + \sum_{i\in\partial a} E_{i\to a}(x_i) \right\}$$

$$E_i = \min_{x_i} \left\{ \sum_{b\in\partial i} \hat{E}_{b\to i}(x_i) \right\}$$

$$E_{ai} = \min_{x_i} \left\{ \hat{E}_{a\to i}(x_i) + E_{i\to a}(x_i) \right\} .$$

3. Application to K-SAT.

We recall the parametrization:

$$E_{i\to a}(s_i) = |h_{i\to a}| - h_{i\to a} s_i$$
$$\hat{E}_{a\to i}(s_i) = |\hat{h}_{a\to i}| - \hat{h}_{a\to i} s_i .$$

Set $J_{ia} = +1$ if edge (ia) is full. It is satisfied by $x_i = +1$, i.e $s_i = -1$. Set $J_{ia} = -1$ if edge (ia) is dashed. Clause a is satisfied by $x_i = 0$, i.e $s_i = +1$. The energy function for each clause is then,

$$E_a(s_{\partial a}) = \prod_{j\in\partial a}\left(\frac{1+s_j J_{ja}}{2}\right).$$

which is zero if at least one $s_j = -J_{ja}$. Keep in mind for later use that s_j SAT's clause a if $s_j = -J_{ja}$ and UNSAT's it if $s_j = J_{ja}$.

Further it will be convenient to set $h_{i\to a} = -u_{i\to a} J_{ia}$ and $\hat{h}_{a\to i} = -\hat{u}_{a\to i} J_{ia}$. The min-sum equations become

$$\begin{cases} \hat{u}_{a\to i} = \prod_{j\in\partial a\setminus i} \vartheta(-u_{j\to a}) \\ u_{j\to a} = \sum_{b\in\partial i\setminus a} J_{bi} J_{ai} \hat{u}_{b\to i} \end{cases}$$

where $\vartheta(x) = 1$ for $x > 0$ and 0 for $x \leq 0$.

Alphabet of message passing equations: for $\hat{u}_{a\to i} \in \{0, 1\}$

and we see that only the $\mathrm{sgn}(u_{j\to a})$ really matters.

So we can replace the min-sum equations by:

$$\begin{cases} \hat{u}_{a\to i} = \prod_{j\in\partial a\setminus i} \vartheta(-u_{j\to a}) \\ u_{j\to a} = \mathrm{sgn}\left\{ \sum_{b\in\partial i\setminus a} J_{ai} J_{bi} \hat{u}_{b\to i} \right\}. \end{cases}$$

Here as always $\mathrm{sgn}\, x = +1$, $x>0$; 0, $x=0$ and 1 $x=-1$.

So for $u_{j\to a}$ we have $u_{j\to a} \in \{-1, 0, +1\}$.

The limit $y \to +\infty$. It is easy to see that

$W_1 = 0$. Indeed

$$W_1 = \min_{s_{\partial a}} \left\{ \prod_{j\in\partial a} \left(\frac{1+s_j J_{ja}}{2} \right) + \sum_{j\in\partial a\setminus i} E_{j\to a}(s_j) \right\}$$

$= 0$ since we choose each s_j which yields $E_{j\to a}(s_j) = 0$

and then $s_i = -J_{ia}$.

For W_2 we have

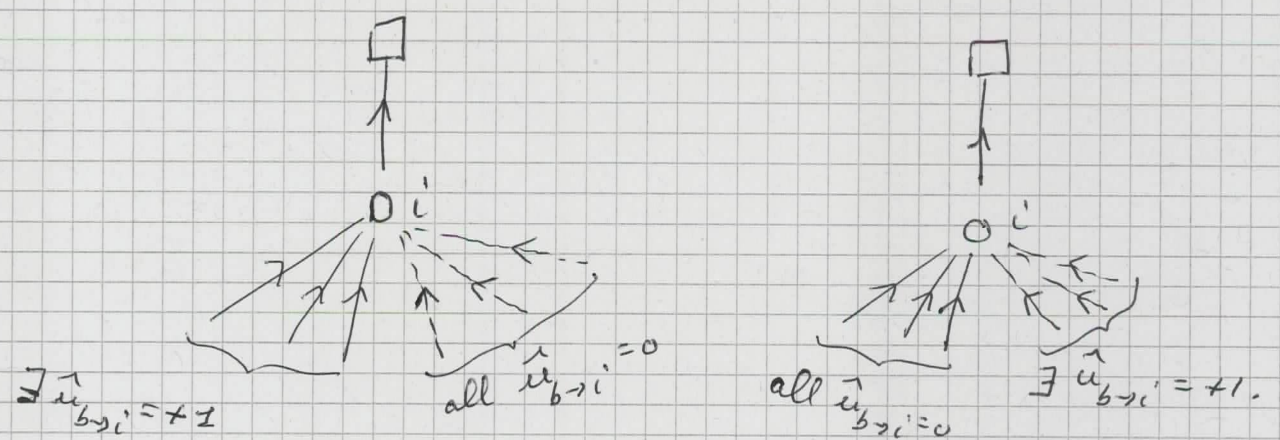
$$W_2 = \min_{s_i} \left\{ \sum_{b\in\partial a\setminus i} \hat{E}_{b\to i}(s_i) \right\}$$

$$= \min_{s_i} \left\{ \sum_{b\in\partial a\setminus i} |\hat{h}_{b\to i}| - s_i \sum_{b\in\partial a\setminus i} \hat{h}_{b\to i} \right\}$$

$$= \sum_{b \in \partial a \setminus i} |\hat{h}_{b\to i}| - \Big|\sum_{b\in\partial a\setminus i} \hat{h}_{b\to i}\Big|$$

$$= \sum_{b\in\partial a\setminus i} |\hat{u}_{b\to i}| - \Big|\sum_{b\in\partial a\setminus i} J_{bi}\, \hat{u}_{b\to i}\Big|$$

In the limit $y \to +\infty$ we have to select only messages for which $W_2 = 0$. This means that for <u>all edges on which $\hat{u}_{b\to i} = 1$</u> we must have <u>all $J_{bi} = +1$</u> or <u>all $J_{bi} = -1$</u>. For edge on which $\hat{u}_{b\to i} = 0$ we may have a mix of signs for J_{bi}.



<u>Surveys.</u> Consider the surveys $\hat{Q}_{a\to i}$. These can parametrized by $\hat{Q}_{a\to i}(1)$ and $\hat{Q}_{a\to i}(0)$; and we must have $\hat{Q}_{a\to i}(1) + \hat{Q}_{a\to i}(0) = 1$. We will set

$$\hat{Q}_{a\to i}(1) = \hat{Q}_{a\to i}.$$

What is the interpretation of $\hat{Q}_{a\to i}$? It is the probability (over pure states) that edge $a\to i$ carries message $\hat{u}_{a\to i} = +1$. This means that edge $a \to i$ carries the energy cost

$$\begin{cases} \hat{E}_{a\to i}(s_i = J_{ia}) = |\hat{u}_{a\to i}| + \hat{u}_{a\to i} J_{ia} \cdot J_{ia} = +2 \\ \hat{E}_{a\to i}(s_i = -J_{ia}) = |\hat{u}_{a\to i}| - \hat{u}_{a\to i} J_{ia} \cdot J_{ia} = 0 . \end{cases}$$

Message $\hat{u}_{a\to i} = +1$,
can be interpreted as a warning that $\overbrace{s_i = J_{ia}}^{\text{choice}}$ (i.e. node i unsat's a) will result in an energy cost.
Alternatively we may say that $\hat{Q}_{a\to i}$ is the probability that a forces i to satisfy it (given the messages it receives from rest of graph).

Consider now $Q_{i\to a}$. These are parametrized by the three possible values of $u_{i\to a} = \{+1, 0, -1\}$. We set

$Q_{i\to a}(+1) = Q^{S}_{i\to a}$; $Q_{i\to a}(-1) = Q^{U}_{i\to a}$ and

$Q_{i\to a}(0) = Q^{*}_{i\to a}$. Let us interpret these probabilities.

- Message $u_{i\to a} = +1$ corresponds to energy costs:

$$\begin{cases} E_{i\to a}(s_i = J_{ia}) = 2 \\ \hat{E}_{i\to a}(s_i = -J_{ia}) = 0 \end{cases}$$

Thus $Q^{S}_{i\to a}$ is the probability that i is forced to satisfy a (given info from rest of graph).

- Message $u_{i\to a} = -1$ corresponds to energy costs:

$$\begin{cases} E_{i\to a}(s_i = J_{ia}) = 0 \\ \hat{E}_{i\to a}(s_i = -J_{ia}) = 2 \end{cases}$$

Thus $Q^{U}_{i\to a}$ is the prob that i is forced to un-satisfy a,

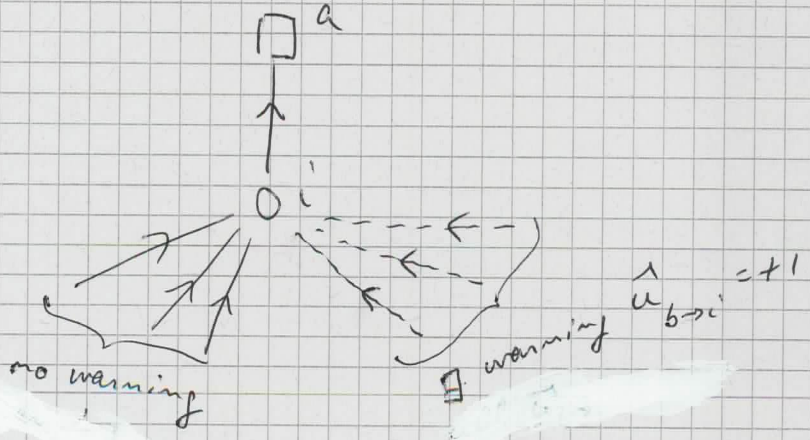
- finally message $u_{i\to a}=0$ corresponds to energy costs $\mathcal{E}_{i\to a}(s_i=J_{ia})=\mathcal{E}_{i\to a}(s_i=-J_{ia})=0$. Thus $Q^*_{i\to a}$ is the prob that either choice of i doesnt cost anything; variable i is "free".

SP equations in the limit $y\to+\infty$.

Consider first equation. In order to have $\hat{u}_{a\to i}=+1$, from the min-sum equation we see that all $u_{j\to a}$ must be equal to -1. Thus

$$\hat{Q}_{a\to i} = \prod_{j\to\partial a\setminus i} Q^U_{i\to a} . \qquad (A)$$

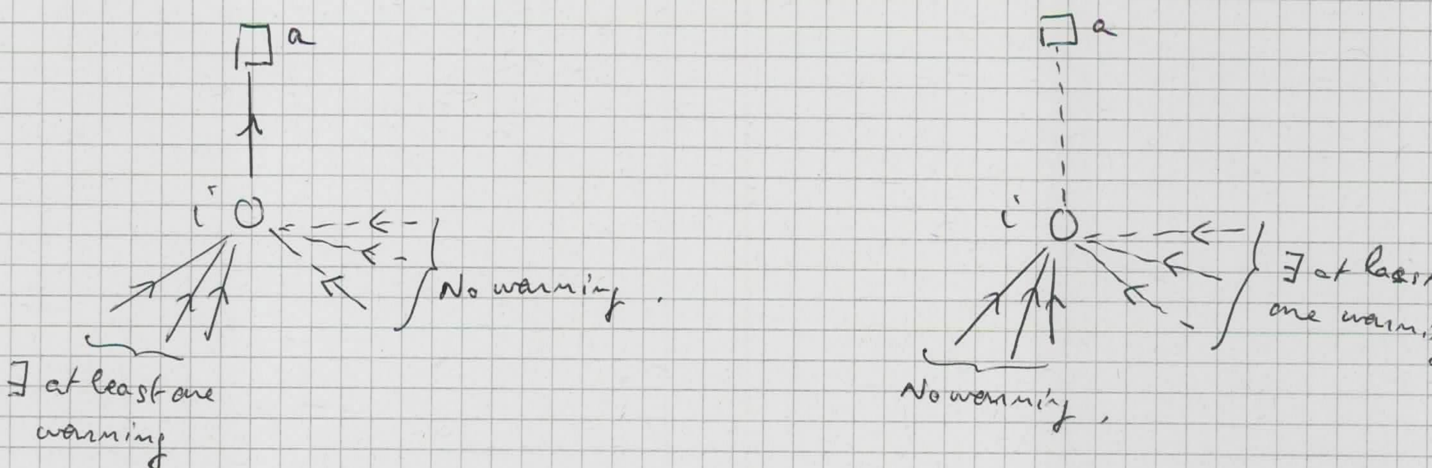
Consider now the second equation. First recall that $y\to+\infty$ imposes the constraint depicted on the previous figure, for example:



We also have to take into account the second min sum equation, say $u_{j\to a}=+1$. Then we must have

$$\mathrm{sgn}\left\{ \sum_{b \in \partial i \setminus a} J_{ai} J_{bi} \hat{u}_{b\to i} \right\} = +1 .$$

then we must have that J_{ai} & J_{bi} have the same sign for edges $b \to i$ with warnings. For example:



This shows that:

$$Q^{s}_{i\to a} \propto \left\{ \prod_{b \in S_{ia}} (1 - \hat{Q}_{i\to a}) \right\} \left\{ 1 - \prod_{b \in U_{ia}} (1 - \hat{Q}_{i\to a}) \right\} . \quad (B)$$

Similarly

$$Q^{u}_{i\to a} \propto \left\{ \prod_{b \in U_{ia}} (1 - \hat{Q}_{i\to a}) \right\} \left\{ 1 - \prod_{b \in S_{ia}} (1 - \hat{Q}_{i\to a}) \right\} . \quad (C)$$

and

$$Q^{*}_{i\to a} \propto \prod_{b \in \partial a \setminus i} (1 - \hat{Q}_{i\to a}) . \quad (D)$$

Finaly one have to normalize by dividing by the sum of the three right hand sides; so that $Q^{s}_{i\to a} + Q^{u}_{i\to a} + Q^{*}_{i\to a} = 1$.

Message passing equations (A), (B), (C), (D) are called Survey Propagation equations. In terms of the surveys the complexity becomes (we can work this out from the analysis of the limit $\lim_{y \to +\infty} \phi_{RSB}(y)$.)

$$\Sigma(\epsilon = 0) = \sum_a \Sigma_a + \sum_i \Sigma_i - \sum_{ai} \Sigma_{ai}$$

$$\Sigma_a = \log \left\{ 1 - \prod_{j \in \partial a} Q^{U}_{j} \right\}$$

$$\Sigma_i = \log \left\{ \prod_{\substack{b \in \partial i \\ \text{undashed} \\ \text{edge}}} (1 - \hat{Q}_{b \to i}) + \prod_{\substack{b \in \partial i \\ \text{dashed} \\ \text{edge}}} (1 - \hat{Q}_{b \to i}) - \prod_{b \in \partial i} (1 - \hat{Q}_{b \to i}) \right\}.$$

$$\Sigma_{ai} = \log \left\{ 1 - Q^{U}_{i \to a} \hat{Q}_{a \to i} \right\}.$$

4. Phase Diagram according to SP.

The averaged complexity over the ensemble of graphs can be computed by a population dynamics method. From the SP equations we see that there is always a trivial fixed point corresponding to

$$\hat{Q} = 0 \quad \text{and} \quad Q^{S} = 0; \; Q^{U} = 0; \; Q^{*} = 1.$$

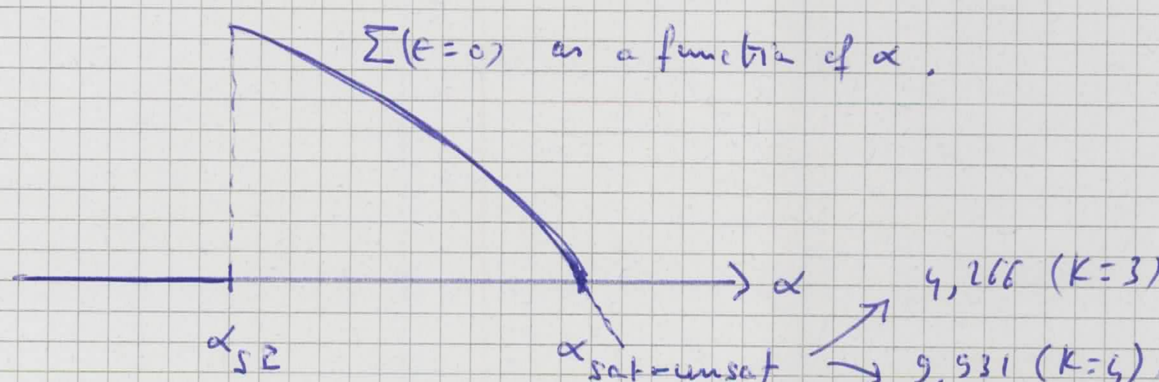
In this phase there are no warning message (i.e all $\hat{u}_{a\to i} = 0$) and variables are "free" to take either values $s_i = \pm 1$. The corresponding complexity is

$$\Sigma(\epsilon = 0) = 0$$

hence the number of clusters / pure states is subexponential. This is identified as a SAT phase with essentially one cluster containing all SAT solutions.

For $\alpha > \alpha_{SP}$ ($K = 3$, $\alpha_{SP} \approx 3{,}93$
$K = 4$, $\alpha_{SP} \approx 8{,}30$).

a non trivial fixed point appears were $\hat{Q} \neq 0$ and Q^s, Q^u, Q^* are not trivial. One find a non trivial average complexity depicted on the figure.



For $\alpha_{SP} < \alpha < \alpha_{sat-unsat}$ the number of clusters is exponentially large. For $\alpha > \alpha_{sat-unsat}$ Σ becomes negative which signals the absence of solutions with $\epsilon = 0$. This is identified as the UNSAT phase. Note that, by computing $\Sigma(\epsilon)$

for $\epsilon \neq 0$ we may find the ground state energy by finding ϵ_{GS} s.t $\Sigma(\epsilon_{GS}) = 0$. This ϵ_{GS} gives the fraction of UNSAT clauses for $\alpha > \alpha_{sat-unsat}$.

Remark 1. The picture outlined here only counts clusters of solutions, regardless of their size. A more refined picture is obtained from the full cavity formalism of lect 12 (sometimes called entropic cavity method) were other thresholds are identified. In particular α_c (condensation threshold) is a threshold s.t for $\alpha \in [\alpha_c, \alpha_{sat-unsat}]$ the number of clusters of maximal size is subexponential.

Remark 2. A cluster of solutions corresponds to the support of a "pure state" at zero temperature. In a pure state one can connect any pair of solutions by a sequence of finite bit flips (staying in solution space). This is a kind of ergodicity within a pure state.

Remark 3. large K expansion are possible :

$$\alpha_{SP} \quad 2^K \frac{\ln K}{K}$$

$$\alpha_{sat-unsat} \quad 2^K \log 2$$

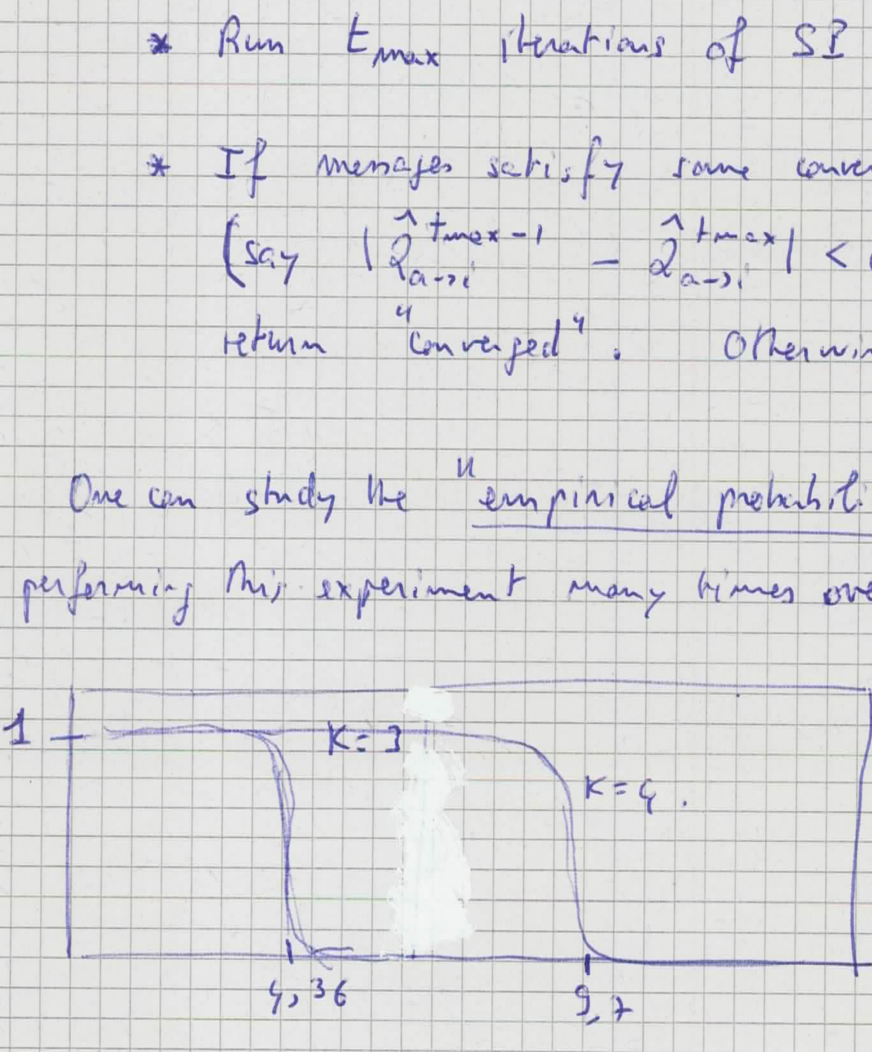
5. An algorithm for finding solutions: SP guided decimation.

We proceed analogously to what we did in lect 8, where we used BP marginals to reduce K-SAT formulas and eventually find a solution. Here we will use SP instead. First we discuss convergence properties of SP-message passing and then present the "SP guided decimation algorithm".

◦ Convergence of SP-message passing.

* Take a graph instance (fix a formula F) of size say $N=10^4$
* Initialize $\hat{Q}_{a\to i}$ randomly in $[0,1]$.
* Run t_{max} iterations of SP (parallel schedule say.) $t_{max} \approx 1000$.
* If messages satisfy some convergence criteria (say $|\hat{Q}^{t_{max}-1}_{a\to i} - \hat{Q}^{t_{max}}_{a\to i}| < \delta = 10^{-2}$) return "converged". Otherwise return "not converged".

One can study the "empirical probability of convergence" by performing this experiment many times over say 100 instances.



complexity of one experiment is $O(N\, t_{max})$.

After SP has converged one can compute the marginals allowing to take decisions for choosing $s_i = \pm 1$. Setting

$$\nu_i(s_i) = \frac{e^{\beta h_i s_i}}{2\cosh h_i \beta} \simeq \exp \beta(h_i s_i - |h_i|) \equiv \exp(-y E_i(s_i))$$

we have that the probability (over clusters) $E_i(s_i = +1) = 0$ & $E_i(s_i = -1) \geq 0$ is
$x_i = 0$
$x_i = 1$

$$Q_i(0) \quad \prod_{\substack{a \in \partial i \\ \text{full edges}}} (1 - \hat{Q}_{a \to i}) \Big/ \Big(1 - \prod_{\substack{a \in \partial i \\ \text{dashed edge}}} (1 - \hat{Q}_{a \to i})\Big).$$

Similarly the prob that $E_i(s_i = 0) > 0$ & $E_i(s_i = +1) = 0$ is

$$Q_i(+1) = \Big(1 - \prod_{\substack{a \in \partial i \\ \text{full edge}}} (1 - \hat{Q}_{a \to i})\Big) \cdot \prod_{\substack{a \in \partial i \\ \text{dashed edge}}} (1 - \hat{Q}_{a \to i}).$$

and the prob that $E_i(s_i = 0) = E_i(s_i = +1) = 0$ is

$$Q_i(*) = \prod_{a \in \partial i} (1 - \hat{Q}_{a \to i}).$$

These marginals are used in the following algorithm.

Note that for $\alpha < \alpha_{SP}$ we have $Q_i(*) = 1$, $Q_i(0) = Q_i(1) = 0$ and we cannot take any decision based on these marginals. One has to use another algorithm (such as BP decimation, Walksat or UCP) ...

- SP - guided decimation.

* Take a fixed instance of size say $N = 10^4$.
* Run SP - message passing. If does not converge return fail. If converges:
* Compute the biases $|Q_i(1) - Q_i(0)|$.
* If all biases are $\leq \delta$ ($\simeq 10^{-1}$) call BP decimation or walksat
* Else fix variable with largest bias to $x_i = +1$ if $Q_i(1) > Q_i(0)$ or $x_i = 0$ if $Q_i(0) > Q_i(1)$
* Reduce formula and recurse (Run SP again).

The complexity of this algo is $O(\underbrace{N t_{max}}_{\text{complexity of each run of SP}}) \cdot \underbrace{N}_{\text{maximal number of iterations or decimation steps}})$.

Note that by decimating a fraction $f \cdot N$ ($f \ll 1$) of variables at a time one will have to recurse $O(f^{-1})$ times so that the complexity is reduced to $O(N t_{max})$. It is believed that $t_{max} = \log N$ is sufficient.

In this algorithm the decimation step drive the formula toward a smaller value of α, so that after a certain number of steps biaises are small and one has to run a local search alg.

- In practice one sees that the empirical probability of finding a solution (repeat the experiment over 100 instances say) is close to 1 for

$$\alpha \lesssim 4,25 \quad (K=3) \qquad \text{and} \qquad \alpha \lesssim 9,6 \quad (K=4)$$

- These values should be compared to the corresponding ones found with BP-decimation :

$$\alpha \lesssim 3,85 \quad (K=3) \qquad \text{and} \qquad \alpha \lesssim 9,25 .$$

- Finaly they can also be compared with the SAT-unsat threshold :

$$\alpha_{sat-unsat} \simeq 4,266 \ (K=3) \qquad \text{and} \qquad \alpha_{sat-unsat} \simeq 9,931 \ (K=4).$$