Proof that Bethe Free Energy correctly predicts the MAP threshold for the BEC and regular codes.

Let C be a fixed (ℓ, r)-regular LDPC code of length n.

We claim that

$$\frac{d H(X \mid Y(\epsilon))}{n \, d\epsilon} = \frac{1}{n} \sum_{i=1}^{n} \Pr\{ \hat{x}_i^{MAP}(Y_{\sim i}) = ? \}$$

Proof:

$$\frac{d H(X \mid Y(\epsilon_1, \dots, \epsilon_n))}{d\epsilon} = \sum_{i=1}^{n} \frac{\partial H(X \mid Y(\epsilon_1, \dots, \epsilon_n))}{\partial \epsilon_i} \Big|_{\epsilon_j = \epsilon}$$

$$\overset{(*)}{=} \sum_{i=1}^{n} \frac{\partial H(X_i \mid Y(\epsilon_1, \dots, \epsilon_n))}{\partial \epsilon_i} \Big|_{\epsilon_j = \epsilon}$$

To see $(*)$ note that

$$H(X \mid Y) = H(X_i \mid Y) + H(X_{\sim i} \mid X_i, Y)$$
$$= H(X_i \mid Y) + H(X_{\sim i} \mid X_i, Y_{\sim i})$$

since the channel is memoryless

But note that $H(X_{\sim i} \mid X_i, Y_{\sim i})$ is not a function of ϵ_i. Further

$$H(X_i \mid Y) = \underbrace{\Pr\{ Y_i = ? \}}_{\epsilon_i} \underbrace{\Pr\{ \hat{x}_i^{MAP}(Y_{\sim i}) = ? \}}_{\text{not a function of } \epsilon_i}$$

Now note that

$$\frac{1}{n}\sum_{i=1}^{n} P_r\{\hat{x}_i^{MAP}(y_{\sim i}) = ?\} \underset{\substack{\uparrow \\ \text{optimality} \\ \text{of MAP}}}{\leq} \frac{1}{n}\sum_{i=1}^{n} P_r\{\hat{x}_i^{BP}(y_{\sim i}) = ?\}$$

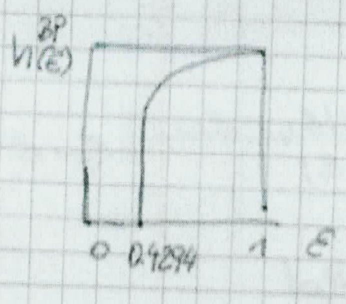
Let us now look closer at the right hand side.

Define

$$h^{BP}(\epsilon) = \lim_{\ell\to\infty} \lim_{n\to\infty} \mathbb{E}_{LDPC}\left[\frac{1}{n}\sum_{i=1}^{n} P_r\{\hat{x}^{BP,\ell}(y_{\sim i}) = ?\}\right]$$

This limit exists and is given by density evolution.

What does $h^{BP}(\epsilon)$ look like?



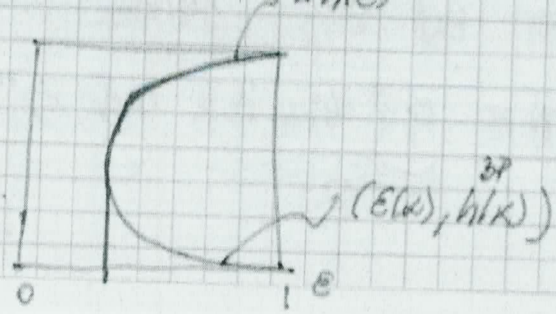
(3, 6) ensemble

In fact we have an explicit characterization.

Define $\quad \epsilon(x) = \dfrac{x}{(1-(1-x)^{r-1})^{\ell-1}}$

$\quad\quad\quad h^{BP}(x) = (1-(1-x)^{r-1})^{\ell}$

Plot $\left(\epsilon(x), h^{BP}(x)\right)_{x=0}^{1}$



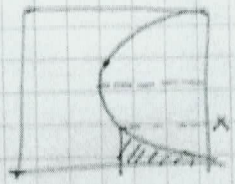
Then the "envelope" of this curve is equal to $h^{BP}(\epsilon)$ as a function of ϵ.

Define the so called trial entropy,

$$P(x) = \int_0^x (1-(1-x)^{r-1})^l \, \epsilon'(x) \, dx$$

$$= x + \frac{1}{r}(1-x)^{r-1}(l + l(r-1)x - rx) - \frac{l}{r}$$

This is equal to the area under the $(\epsilon(x), h^{BP}(x))$ curve.



Note that $P(x_2) - P(x_1)$, where $x^{BP} \le x_1 \le x_2$ is ≥ 0 and equal to the area under the $h^{BP}(\epsilon)$ curve. In particular, define the area threshold ϵ^A as the unique number $\ge \epsilon^{BP}$ so that, $\epsilon^A = \epsilon(x^A)$ where

$$P(1) - P(x^A) = r(l,r) = 1 - \frac{l}{r}$$

Note that $P(1) = 1 - \frac{l}{r}$ (just check). So we are looking for x^A so that $P(x^A) = 0$ and $\epsilon(x^A)$ is called the area threshold.

Now consider

$$\limsup_{n\to\infty} \mathbb{E}_{LDPC}\Big[\underbrace{\int_{\tilde{\epsilon}}^{1} \frac{1}{n}\sum_{i=1}^{n} P_r\{\hat{X}_i^{MAP}(Y_{\sim i}^{\epsilon}) = ?\}\, d\epsilon}_{\text{by definition } = \frac{1}{n}\left(H(X\mid Y(\epsilon=1)) - H(X\mid Y(\epsilon=\tilde{\epsilon}))\right)}$$

$$= \underbrace{r(\ell, r)}_{\substack{\text{this needs} \\ \text{some proof that} \\ \text{this is equal to} \\ 1-\frac{\ell}{r}}} - \liminf_{n\to\infty} \mathbb{E}_{LDPC}\left[\frac{1}{n} H(X\mid Y(\epsilon=\tilde{\epsilon}))\right]$$

$$= \limsup_{n\to\infty} \int_{\tilde{\epsilon}}^{1} \mathbb{E}\left[\frac{1}{n} P_r\{\hat{X}_i^{MAP}(Y_{\sim i}^{\epsilon}) = ?\}\right] d\epsilon$$

↑ Fubini; non-negative function

$$\leq \int_{\tilde{\epsilon}}^{1} \limsup_{n\to\infty} \mathbb{E}\left[\frac{1}{n} P_r\{\hat{X}_i^{MAP}(Y_{\sim i}) = ?\}\right] d\epsilon$$

↑ Fatou-Lebesgue; function is bounded

$$\leq \int_{\tilde{\epsilon}}^{1} \lim_{\ell\to\infty} \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n} P_r\{\hat{X}_i^{BP,\ell}(Y_{\sim i}) = ?\}\right] d\epsilon$$

We conclude; for ~~all~~ $x^{BP} \le x \le 1$;

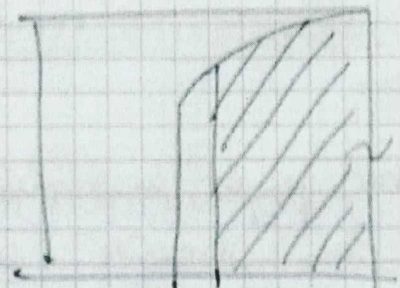
$$1-\frac{\epsilon}{r} - \liminf_{n\to\infty} \mathbb{E}_{LDPC}\left[\frac{1}{n} H(X \mid Y(\epsilon=\epsilon(x)))\right]$$

$$\le 1-\frac{\epsilon}{r} - P(x)$$ or equivalently

$$\liminf_{n\to\infty} \mathbb{E}_{LDPC}\left[\frac{1}{n} H(X \mid Y(\epsilon=\epsilon(x)))\right] \ge P(x)$$

We conclude that $\epsilon^{MAP} \ge \epsilon^A = \epsilon(x^A)$.

area threshold



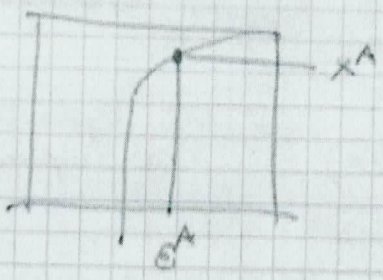
$1-\frac{\epsilon}{r}$

$\epsilon^{MAP} \le \epsilon^A$

How do we see the opposite inequality?

Let us now quickly outline how one can show that up to ϵ^A a MAP decoder can decode with high probability.

We proceed as follows. Pick $\varepsilon \leq \varepsilon^{A}$.
Start with BP decoding. The decoder will get stuck.



(*) Expurgate from the "set of equations" (the bipartite graph) all variables which are known at this point.

(*) this leaves a residual graph
(*) conditioned on the degree distribution this graph is still random

(*) prove that whp this graph has full rank, i.e, a MAP decoder can decode (or essentially full rank); this shows that

$$\frac{1}{n} H(X \mid Y(\varepsilon)) \to 0$$

To show the last part we can use the Markov inequality.

Let $N(C)$ be the number of solutions to the system of remaining equations.

It is not too hard to compute

$$E_{LDPC}[N(C)]$$

since we can average over the ensemble.
Now note that $N(C)$ is integer valued. Hence

$$\Pr\{N(C) \geq 1\} \leq E_{LDPC}[N(C)].$$

Note: This would not work if we had started with the original matrix! It is crucial that we first use BP!!