ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 | Statistical Physics for Communication and Computer Science
Homework 6 | March 31, 2011, INR 113 - 9:15-11:00

Problem 1 (Gallager Algorithm A)**.** In class we discussed the BP algorithm which is the "locally optimal" message-passing algorithm. One of its downsides in a practical application is that it requires the exchange of real numbers. Hence, in any implementation messages are quantized to a fixed number of bits. One way to think of such a quantized algorithm is that the message represents an "approximation" of the underlying message that BP would have sent.

Assume that we are limited to exchange messages consisting of a single bit. Recall that for BP a positive message means that our current estimate of the associated bit is $+1$, whereas a negative message means that our current estimate is -1 (the magnitude of the BP message conveys our certainty). So we can think of a message-passing algorithm which is limited to exchange messages consisting of a single bit, as exchanging only the sign of their estimate.

The best known such algorithm (and historically also the oldest) is Gallager's algorithm A. It has the following message passing rules.

We assume that the codewords and the received word have components in $\{0, 1\}$.

(i) *Initialization:* In the first iteration send out the received bits along all edges incident to a variable node.

(ii) *Check Node Rule:* At a check node send out along edge e the XOR of the incoming messages (not counting the incoming message along edge e).

(iii) *Variable Node Rule:* At a variable node. Send out the received value along edge e unless all incoming messages (not counting the incoming message on edge e) all agree in their value. Then send this value.

Assume that transmission takes place over the BSC(p) and that we are using a $(3, 6)$-regular Gallager ensemble. Write down the density evolution equations for the Gallager algorithm A. What is the threshold for this combination?

Problem 2 (Density Evolution via Population Dynamics)**.** In class we have seen the density evolution (DE) for transmission over the BEC. This was relatively easy since in this case the "densities" are in fact numbers (erasure probabilities). For general channels, DE is more involved since it really involves the evolution of densities. These are the densities of messages which you would see at the various iterations if you implemented the BP message-passing decoder on an infinite ensemble for a fixed number of iterations.

An quick and dirty way of implementing DE for general channels is by means of a population dynamics approach. Here is how this works. Assume that transmission takes place over a given BMS channel and that we are using the (l, r)-regular Gallager ensemble. Pick a population size N. The larger N the more accurate will be your result but the slower it will be.

(i) Pick an *initial* population, call it $\mathcal{V}_0$. This set consists of N iid log-likelihoods associated to the given BMS channel, assuming that the transmitted bit is 1 (we are using spin notation here). More precisely, each sample is created in the following way. Sample Y according to $p(y \mid x = 1)$. Compute the corresponding log-likelihood value, call it L.

(ii) Starting with $\ell = 1$, where ℓ denotes the iteration number, compute now the densities corresponding to the ℓ-th iteration in the following way.

(iii) To compute $\mathcal{C}_\ell$ proceed as follows. Create N samples iid in the following way. For each sample, call it Y, pick $r - 1$ samples from $\mathcal{C}_{\ell-1}$ with repetitions. Let these samples be named $X_1, \ldots, X_{r-1}$. Compute $Y = 2\tanh^{-1}(\prod_{i=1}^{r-1} \tanh(X_i/2))$. Note, these are exactly the message-passing rules at a check node.

(iv) To compute $\mathcal{V}_\ell$ proceed as follows. Create N samples iid in the following way. For each sample, call it Y, pick $l - 1$ samples from $\mathcal{C}_\ell$ with repetitions. Let these samples be named $X_1, \ldots, X_{l-1}$. Further, pick a sample from $\mathcal{V}_0$, call it C. Compute $Y = C + \sum_{i=1}^{l-1} X_i$. Note, these are exactly the message-passing rules at a variable node.

We think now of each set $\mathcal{V}_\ell$ and $\mathcal{C}_\ell$ as a sample of the corresponding distribution. E.g., in order to construct this distribution approximately we might use a histogram applied to the set. Recall, that we assume here the all-zero codeword assumption. Hence, in order to see whether this experiments corresponds to a successful decoding, we need to check whether in $\mathcal{V}_\ell$ all samples have positive sign and magnitude which converges (in ℓ) to infinity.

Implement the population dynamics approach for transmission over the BAWGNC(σ) channel using the $(3, 6)$-regular Gallager ensemble. Estimate the threshold using this method. Plot the threshold on the same plot as the simulation results which you performed for your last homework. Hopefully this vertical line, indicating the threshold, is somewhere around where the error probability curves show a sharp drop-off.