ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 11 Statistical Physics for Communication and Computer Science
Homework 11 May 12, 2011, INR 113 - 9:15-11:00

Suppose you want to compute the free energy of K-SAT. First you should recognize that the partition function of K-SAT

$$Z = \sum_{\underline{x}} \prod_a f_a(x_{\partial a})$$

counts the number of solutions of an instance. Here $\underline{x} = (x_1, \cdots, x_N)$ and f_a the appropriate clause functions that depend on the K variables participating to a. In the SAT phase the number of solutions is exponential in N. Then the average free energy[1] defined as $f = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}(\ln Z)$ ($\mathbb{E}$ is the expectation with respect to the graph ensemble defined in lecture 1) is the average rate function for the number of solutions. We expect that this function is positive in the SAT phase $\alpha < \alpha_s$ and that it becomes negative beyond α_s. Thus a computation of the free energy may give us a value for the SAT-UNSAT threshold. However we dont know how to compute this function exactly.

In this exercise you will compute the average Bethe free energy for $K = 3$. You will see that the threshold predicted by this computation is around $\alpha_B \approx 4.677$. However the upper bound that you proved in handout 10 yields $\alpha_s < 4.667$ (and there exist even better bounds). Thus the average Bethe free energy cannot be exact for K-SAT (unlike the situation in coding). In fact the cavity method predicts that $\alpha_s \approx 4.26$. Concerning the Bethe free energy it predicts that it should be equal to the true free energy for $\alpha < 3.86$.

Problem 1 (Bethe free energy for K-SAT). Recall from lecture 10 that the Bethe free energy is the functional

$$F_{Bethe}[\nu, \hat{\nu}] = \sum_a F_a + \sum_i F_i - \sum_{ai} F_{ai}$$

with

$$F_a = \ln\Big\{ \sum_{x_{\partial a}} f_a(x_{\partial a}) \prod_{i\in\partial a} \nu_{i\to a}(x_i) \Big\}$$

$$F_i = \ln\Big\{ \sum_{x_i} \prod_{b\in\partial i} \hat{\nu}_{b\to i}(x_i) \Big\}$$

$$F_{ai} = \ln\Big\{ \sum_{x_i} \nu_{i\to a}(x_i)\hat{\nu}_{a\to i}(x_i) \Big\}$$

a) Write this expression in terms of the $\zeta_{a\to i}$ and $\hat{\zeta}_{i\to a}$ defined in handout and lecture 8.

b) Show that the stationary point equations are precisely the BP fixed point equations given in handout and lecture 8 in terms of $\zeta_{a\to i}$ and $\hat{\zeta}_{i\to a}$.

Problem 2 (Average Bethe free energy). Proceed as in the class and define random variables ζ and $\hat{\zeta}$ corresponding to the messages. Their distributions satisfy a corresponding fixed point equation. Define an average form of the Bethe free energy, where the average

[1] Note that here it is natural to define free energy with the "plus" sign

is over the graph ensemble and over the random variables ζ and $\hat{\zeta}$. Hint: recall that check degrees are all fixed to K and node degrees are Poisson with mean αK. Moreover the number of dashed (resp full) edges incident to a variable node is Poisson with mean $\alpha K/2$ (resp. $\alpha K/2$). From now on take $K = 3$

a) Solve the distributional equations for ζ and $\hat{\zeta}$ by the method of population dynamics. Recall that you used this method already for solving DE for the BAWGNC in hand out 6, problem 2. Here proceed analogously.

b) Compute then the average Bethe free energy and plot it as a function of α. Observe if the threshold predicted by this method is $\alpha_B = 4.677$?