Solution de la série 6
Traitement Quantique de l'Information

Exercice 1

1. One has to show that $\langle B_{x,y}| B_{x',y'}\rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases :

$$\begin{aligned}\langle B_{00}| B_{00}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle)\\ &= \frac{1}{2}(\langle 00| 00\rangle + \langle 00| 11\rangle + \langle 11| 00\rangle + \langle 11| 11\rangle).\end{aligned}$$

Now we have

$$\begin{aligned}\langle 00| 00\rangle = \langle 0| 0\rangle \langle 0| 0\rangle = 1, \langle 00| 11\rangle = \langle 0| 1\rangle \langle 0| 1\rangle = 0,\\ \langle 11| 00\rangle = \langle 1| 0\rangle \langle 1| 0\rangle = 0, \langle 11| 11\rangle = \langle 1| 1\rangle \langle 1| 1\rangle = 1.\end{aligned}$$

Thus we get that $\langle B_{00}| B_{00}\rangle = \frac{1}{2}(1+0+0+1) = 1$. Now let us consider

$$\begin{aligned}\langle B_{00}| B_{01}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle)\\ &= \frac{1}{2}(\langle 00| 01\rangle + \langle 00| 10\rangle + \langle 11| 01\rangle + \langle 11| 10\rangle)\\ &= \frac{1}{2}(0+0+0+0) = 0.\end{aligned}$$

2. The proof is by contradiction. Suppose there exist a_1, b_1 and a_2, b_2 such that

$$|B_{00}\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle).$$

Then we must have

$$\frac{1}{2}(|00\rangle + |11\rangle) = a_1a_2 |00\rangle + a_1b_2 |01\rangle + b_1a_2 |10\rangle + a_2b_2 |11\rangle .$$

Since the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis one has

$$\frac{1}{2} = a_1a_2,\ \frac{1}{2} = b_1b_2,\ a_1b_2 = 0,\ b_1a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, B_{00} is entangled.

3. We have

$$\begin{aligned}|\gamma\rangle \otimes |\gamma\rangle &= (\cos(\gamma)\,|0\rangle + \sin(\gamma)\,|1\rangle) \otimes (\cos(\gamma)\,|0\rangle + \sin(\gamma)\,|1\rangle) \\ &= \cos^2(\gamma)\,|00\rangle + \cos(\gamma)\sin(\gamma)\,|01\rangle + \sin(\gamma)\cos(\gamma)\,|10\rangle + \sin^2(\gamma)\,|11\rangle\,.\end{aligned}$$

Similarly,

$$|\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = \cos^2(\gamma_\perp)\,|00\rangle + \cos(\gamma_\perp)\sin(\gamma_\perp)\,|01\rangle + \sin(\gamma_\perp)\cos(\gamma_\perp)\,|10\rangle + \sin^2(\gamma_\perp)\,|11\rangle\,.$$

A picture shows that $\cos(\gamma_\perp) = -\sin(\gamma)$ and $\sin(\gamma_\perp) = \cos(\gamma)$ (this also allows to check that $\langle\gamma|\,\gamma_\perp\rangle = 0$). Therefore, $\cos^2(\gamma_\perp) = \sin^2(\gamma)$, $\sin^2(\gamma_\perp) = \cos^2(\gamma)$ and $\cos(\gamma_\perp)\sin(\gamma_\perp) = -\cos(\gamma)\sin(\gamma)$. We find that

$$|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = (\cos^2(\gamma) + \sin^2(\gamma))\,|00\rangle + (\sin^2(\gamma) + \cos^2(\gamma))\,|11\rangle\,,$$

and the terms $|01\rangle$ and $|10\rangle$ cancel. Finally,

$$\frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle\,.$$

4. From the rule for the tensor product

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix},$$

we get for the basis states

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus,

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix},$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Exercice 2

1. By definition of the tensor product :

$$(H \otimes I)\,|x\rangle \otimes |y\rangle = H\,|x\rangle \otimes I\,|y\rangle = H\,|x\rangle \otimes |y\rangle\,.$$

Also, one can use that $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ to show that always

$$H\,|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle).$$

Thus,

$$(H \otimes I)\,|x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x\,|1\rangle \otimes |y\rangle).$$

Note that this state is not entangled. Indeed $(H \otimes I)\,|x\rangle \otimes |y\rangle = H|x\rangle \otimes |y\rangle$ which is a tensor product state.
Now we apply *'CNOT'*. By linearity, we can apply it to each term separately. Thus,

$$\begin{aligned}(CNOT)(H \otimes I)\,|x\rangle \otimes |y\rangle &= \frac{1}{\sqrt{2}}((CNOT)\,|0\rangle \otimes |y\rangle + (-1)^x(CNOT)\,|1\rangle \otimes |y\rangle)\\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |y\rangle + (-1)^x\,|1\rangle \otimes |y \oplus 1\rangle)\\ &= |B_{xy}\rangle\,.\end{aligned}$$

2. Let us first start with $H \otimes I$. We use the rule

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \otimes \begin{pmatrix}e & f\\ g & h\end{pmatrix} = \begin{pmatrix}ae & af & be & bf\\ ag & ah & bg & bh\\ ce & cf & de & df\\ cg & ch & dg & dh\end{pmatrix},$$

Thus we have

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}.$$

For (CNOT), we use the definition :

$$(CNOT)\left|x\right\rangle\otimes\left|y\right\rangle=\left|x\right\rangle\otimes\left|y\oplus x\right\rangle,$$

which implies that the matrix elements are

$$\left\langle x'y'\right|CNOT\left|xy\right\rangle=\left\langle x',y'|x,y\otimes x\right\rangle=\left\langle x'|\,x\right\rangle\left\langle y'|\,y\oplus x\right\rangle=\delta_{xx'}\delta_{y\oplus x,y'}.$$

We obtain the following table with columns xy and rows $x'y'$:

	00	01	10	11
00	1	0	0	0
01	0	1	0	0
10	0	0	0	1
11	0	0	1	0

For the matrix product $(CNOT)(H\otimes I)$, we find that

$$\begin{aligned}(CNOT)H\otimes I&=\frac{1}{\sqrt{2}}\begin{pmatrix}I & 0\\ 0 & X\end{pmatrix}\begin{pmatrix}I & I\\ I & -I\end{pmatrix}\\ &=\frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ X & -X\end{pmatrix},\end{aligned}$$

where $X=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Thus,

$$(CNOT)(H\otimes I)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}.$$

One can check that for example $\left|B_{00}\right\rangle=(CNOT)(H\otimes I)\left|0\right\rangle\otimes\left|0\right\rangle$. Finally to check the unitarity, we have to check that $UU^\dagger=U^\dagger U=I$ for $U=H\otimes I$, $CNOT$ and $(CNOT)(H\otimes I)$. We leave this to the reader.

3. Let $U=(CNOT)(H\otimes I)$. We have

$$\left|B_{xy}\right\rangle=U\left|x\right\rangle\otimes\left|y\right\rangle,\left\langle B_{x'y'}\right|=\left\langle x'\right|\otimes\left\langle y'\right|U^\dagger.$$

Thus,

$$\begin{aligned}\left\langle B_{x'y'}|\,B_{xy}\right\rangle&=\left\langle x'\right|\otimes\left\langle y'\right|U^\dagger U\left|x\right\rangle\otimes\left|y\right\rangle\\ &=\left\langle x'\right|\otimes\left\langle y'\right|I\left|x\right\rangle\otimes\left|y\right\rangle\\ &=\left\langle x'|\,x\right\rangle\left\langle y'|\,y\right\rangle=\delta_{xx'}\delta_{yy'}.\end{aligned}$$

Exercice 3

1. The possible outcomes of the measurement are simply the basis states. Let us compute the probability that the first basis state $|\alpha\rangle \otimes |\beta\rangle$ is the outcome. According to the measurement principle :

$$\text{Prob}(\alpha, \beta) = \Big|(\langle\alpha| \otimes \langle\beta|)(|B_{00}\rangle)\Big|^2$$

Using $|B_{00}\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle \otimes |\alpha\rangle + \frac{1}{\sqrt{2}}|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle$ (see exercise 1 and choose $\gamma = \alpha$) we get

$$\begin{aligned}\text{Prob}(\alpha, \beta) =& \frac{1}{2}|\langle\alpha|\alpha\rangle\langle\beta|\alpha\rangle + \langle\alpha|\alpha_\perp\rangle\langle\beta|\alpha_\perp\rangle|^2 \\ =& \frac{1}{2}(\cos(\alpha - \beta))^2\end{aligned}$$

For the three other probabilities we have

$$\begin{aligned}\text{Prob}(\alpha, \beta_\perp) &= \frac{1}{2}(\cos(\alpha - \beta_\perp))^2 = \frac{1}{2}(\sin(\alpha - \beta))^2 \\ \text{Prob}(\alpha_\perp, \beta) &= \frac{1}{2}(\cos(\alpha_\perp - \beta))^2 = \frac{1}{2}(\sin(\alpha - \beta))^2 \\ \text{Prob}(\alpha_\perp, \beta_\perp) &= \frac{1}{2}(\cos(\alpha_\perp - \beta_\perp))^2 = \frac{1}{2}(\cos(\alpha - \beta))^2\end{aligned}$$

2. In her lab Alice observes $|\alpha\rangle$ ou $|\alpha_\perp\rangle$. Using the results above (with $(\cos^2 + \sin^2 = 1)$ we find the probabilities

$$\text{Prob}(\alpha) = \text{Prob}(\alpha, \beta) + \text{Prob}(\alpha, \beta_\perp) = \frac{1}{2}$$

et

$$\text{Prob}(\alpha_\perp) = \text{Prob}(\alpha_\perp, \beta) + \text{Prob}(\alpha_\perp, \beta_\perp) = \frac{1}{2}$$

De même Bob dans son labo observe $|\beta\rangle$ ou $|\beta_\perp\rangle$ avec probabilités 1/2. So from the perspective of Alice and Bob each quantum bit is completely random !

3. *First only Alice measures.* The resulting states are calculated by acting with the projectors on $|B_{00}\rangle$:

$$(|\alpha\rangle\langle\alpha| \otimes I)|B_{00}\rangle, \qquad (|\alpha_\perp\rangle\langle\alpha_\perp| \otimes I)|B_{00}\rangle$$

Using the formula of exercise 1 for $\gamma = \alpha$ we find the two states

$$\frac{1}{\sqrt{2}}|\alpha\rangle \otimes |\alpha\rangle, \qquad \frac{1}{\sqrt{2}}|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle$$

Since we should normalise the states we must discard the $1/\sqrt{2}$ in these formulas. The probabilities are

$$\Big|(\langle\alpha| \otimes \langle\alpha|)(|B_{00}\rangle)\Big|^2 = \frac{1}{2}, \qquad \Big|(\langle\alpha_\perp| \otimes \langle\alpha_\perp|)(|B_{00}\rangle)\Big|^2 = \frac{1}{2}$$

Now Bob measures. Thus the states just obtained after Alice's measurement are projected with the projectors $I \otimes |\beta\rangle\langle\beta|$ or $I \otimes |\beta_\perp\rangle\langle\beta_\perp|$.

– If after Alice's measurement the state is $|\alpha\rangle \otimes |\alpha\rangle$ (occurs with prob 1/2) when Bob measures the state becomes (proportional to)

$$(I\otimes|\beta\rangle\langle\beta|)(|\alpha\rangle\otimes|\alpha\rangle) = \langle\beta|\alpha\rangle\, |\alpha\rangle\otimes|\beta\rangle, \qquad (I\otimes|\beta_\perp\rangle\langle\beta_\perp|)(|\alpha\rangle\otimes|\alpha\rangle) = \langle\beta_\perp|\alpha\rangle\, |\alpha\rangle\otimes|\beta_\perp\rangle$$

with probabilities

$$\frac{1}{2}|\langle\alpha| \otimes \langle\beta|\alpha\rangle \otimes |\alpha\rangle|^2 = \frac{1}{2}(\cos(\alpha-\beta))^2, \qquad \frac{1}{2}|\langle\alpha| \otimes \langle\beta_\perp|\alpha\rangle \otimes |\alpha\rangle|^2 = \frac{1}{2}(\sin(\alpha-\beta))^2$$

– If after Alice's measurement the state is $|\alpha_\perp\rangle \otimes |\alpha_\perp\rangle$ (occurs with prob 1/2) when Bob measures the state becomes (proportional to)

$$(I\otimes|\beta\rangle\langle\beta|)(|\alpha_\perp\rangle\otimes|\alpha_\perp\rangle) = \langle\beta|\alpha_\perp\rangle\, |\alpha_\perp\rangle\otimes|\beta\rangle, \qquad (I\otimes|\beta_\perp\rangle\langle\beta_\perp|)(|\alpha_\perp\rangle\otimes|\alpha_\perp\rangle) = \langle\beta_\perp|\alpha_\perp\rangle\, |\alpha_\perp\rangle\otimes|\beta_\perp\rangle$$

with probabilities

$$\frac{1}{2}|\langle\alpha_\perp|\otimes\langle\beta|\alpha_\perp\rangle\otimes|\alpha_\perp\rangle|^2 = \frac{1}{2}(\sin(\alpha-\beta))^2, \qquad \frac{1}{2}|\langle\alpha_\perp|\otimes\langle\beta_\perp|\alpha_\perp\rangle\otimes|\alpha_\perp\rangle|^2 = \frac{1}{2}(\cos(\alpha-\beta))^2$$

4. The previous question implies that when Alice does the measurement first and Bob after :

   – Alice got the result $|\alpha\rangle$ or $|\alpha_\perp\rangle$ with prob 1/2.

   – Bob got in his lab the result $|\beta\rangle$ with probability

   $$\frac{1}{2}(\cos(\alpha-\beta))^2 + \frac{1}{2}(\sin(\alpha-\beta))^2 = \frac{1}{2}$$

   or the result $|\beta_\perp\rangle$ with probability

   $$\frac{1}{2}(\sin(\alpha-\beta))^2 + \frac{1}{2}(\cos(\alpha-\beta))^2 = \frac{1}{2}$$

5. Summarising, this exercise has shown that the observations of Alice and Bob in each lab are the same wether the measurements are done simultaneously or in a series. With no communication between Alice and Bob the net result is :

   – Alice chooses a measurement basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ and gets the outcomes $|\alpha\rangle$ or $|\alpha_\perp\rangle$ with probability 1/2 ;

   – Bob chooses a measurement basis $\{|\beta\rangle, |\beta_\perp\rangle\}$ and gets the outcomes $|\beta\rangle$ or $|\beta_\perp\rangle$ with probability 1/2.

   With no communication the entanglement (intrication) is never detectable by local operations. Quantum bits in each separate lab appear to Alice and Bob as completely disordered or random.