Solution de la série 4
Traitement Quantique de l'Information

**Exercice 1** *Heisenberg Uncertainty Principle.*

1. Let $|\phi_\lambda\rangle = (A + i\lambda B)|\psi\rangle$. For $\lambda \in \mathbb{R}$, let us define $f(\lambda) = \langle\psi_\lambda|\,\psi_\lambda\rangle$. It is clear that for any $\lambda \in \mathbb{R}$, $f(\lambda) \geq 0$. We also have

$$\begin{aligned} f(\lambda) &= \langle\psi|(A^\dagger - i\lambda^* B^\dagger)(A + i\lambda B)|\psi\rangle = \langle\psi|(A - i\lambda B)(A + i\lambda B)|\psi\rangle \\ &= \langle\psi| A^2 |\psi\rangle + \lambda^2 \langle\psi| B^2 |\psi\rangle + i\lambda \langle\psi|(AB - BA)|\psi\rangle \\ &= \langle\psi| A^2 |\psi\rangle + \lambda^2 \langle\psi| B^2 |\psi\rangle + \lambda \langle\psi| i[A, B] |\psi\rangle, \end{aligned}$$

where we used the Hermitian property of $A$ and $B$ and the fact that $\lambda \in \mathbb{R}$, thus $\lambda = \lambda^*$. First one can simply check that the operator $i[A, B]$ is a Hermitian operator so the last term $\langle\psi| i[A, B] |\psi\rangle$ is real-valued so $f(\lambda)$ is real-valued (this was already clear). We see that $f(\lambda)$ is a second order polynomial in $\lambda \in \mathbb{R}$. As it is non-negative for every value of $\lambda$, it results that its discriminant must be negative or zero (for a quadratic equation $ax^2 + bx + c = 0$ the discriminant is defined by $\Delta = b^2 - 4ac$). Hence, we get

$$|\langle\psi| [A, B] |\psi\rangle|^2 \leq 4 \langle\psi| A^2 |\psi\rangle \langle\psi| B^2 |\psi\rangle .$$

As we assumed that the operators have zero mean, $\langle\psi| A |\psi\rangle = \langle\psi| B |\psi\rangle = 0$, we obtain that

$$\Delta A\, \Delta B \geq \sqrt{\frac{|\langle\psi| [A, B] |\psi\rangle|^2}{4}} = \frac{|\langle\psi| [A, B] |\psi\rangle|}{2}$$

2. This time we will use Cauchy-Schwartz inequality that $|\langle a|\, b\rangle|^2 \leq \langle a|\, a\rangle \langle b|\, b\rangle$ for any vector $a$ and $b$ in any Hilbert space. We also use the following inequality that for any two complex numbers $x$ and $y$, $|x - y|^2 \leq 2(|x|^2 + |y|^2)$ which one can simply prove.

$$\begin{aligned} |\langle\psi| [A, B] |\psi\rangle|^2 = |\langle\psi| AB |\psi\rangle - \langle\psi| BA |\psi\rangle|^2 &\overset{(a)}{\leq} 2(|\langle\psi| AB |\psi\rangle|^2 + |\langle\psi| BA |\psi\rangle|^2) \\ &\overset{(b)}{\leq} 2(\langle\psi| AA^\dagger |\psi\rangle \langle\psi| B^\dagger B |\psi\rangle + \langle\psi| BB^\dagger |\psi\rangle \langle\psi| A^\dagger A |\psi\rangle) \\ &\overset{(c)}{=} 4 \langle\psi| A^2 |\psi\rangle \langle\psi| B^2 |\psi\rangle = 4(\Delta A)^2(\Delta B)^2, \end{aligned}$$

where $(a)$ follows from the inequality for complex numbers just mentioned, $(b)$ follows by applying the Cauchy-Schwartz inequality and $(c)$ follows from the Hermitian property of $A$ and $B$.

3. For $\psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle)$ and $A = X$ and $B = Z$, we have

$$\begin{aligned}
\bar{A} = \langle\psi|\,A\,|\psi\rangle &= \frac{1}{2}(1 \quad -i)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 0,\\
\bar{A^2} = \langle\psi|\,A^2\,|\psi\rangle &= \frac{1}{2}(1 \quad -i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 1,\\
\bar{B} = \langle\psi|\,B\,|\psi\rangle &= \frac{1}{2}(1 \quad -i)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 0,\\
\bar{B^2} = \langle\psi|\,B^2\,|\psi\rangle &= \frac{1}{2}(1 \quad -i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 1,
\end{aligned}$$

where for an operator $C$, $\bar{C}$ denotes the average of the operator in state $|\psi\rangle$. Therefore, we get

$$(\Delta A)^2 = \langle\psi|\,A^2\,|\psi\rangle - (\langle\psi|\,A\,|\psi\rangle)^2 = 1, (\Delta B)^2 = \langle\psi|\,B^2\,|\psi\rangle - (\langle\psi|\,B\,|\psi\rangle)^2 = 1.$$

We also know that the Pauli matrices satisfy the identity $[X, Z] = 2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Thus, we have

$$\langle\psi|\,[A, B]\,|\psi\rangle = 2\frac{1}{2}(1 \quad -i)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = -2i,$$

which satisfies the uncertainty principle

$$(\Delta A)^2(\Delta B)^2 = 1 \geq \frac{|2i|^2}{4} = \frac{|\,\langle\psi|\,[A, B]\,|\psi\rangle\,|^2}{4}.$$

In particular, in this case the uncertainty inequality turns out to be an equality.

4. In this exercise, the state of a particle is represented by a function of coordinate variable $\psi(x)$. We also know how $\hat{x}$ and $\hat{p}$ operators transform this state function. To find the commutator of $\hat{x}$ and $\hat{p}$, we take an arbitrary state function $\phi(x)$ and investigate how $[\hat{x}, \hat{p}]$ operates on it. Specifically we have

$$\begin{aligned}
&(\hat{p}\hat{x})\phi(x) = \hat{p}(\hat{x}\phi(x)) = \hat{p}(x\phi(x)) = -i\hbar\frac{d}{dx}(x\phi(x)) = -i\hbar(\phi(x) + x\frac{d}{dx}\phi(x)),\\
&(\hat{x}\hat{p})\phi(x) = \hat{x}(-i\hbar\frac{d}{dx}\phi(x)) = -i\hbar x\frac{d}{dx}\phi(x),
\end{aligned}$$

which implies that $[\hat{x}, \hat{p}]\phi(x) = i\hbar\phi(x)$. As $\phi$ was an arbitrary function, it results that $[\hat{x}, \hat{p}] = i\hbar$.