Exercice 1 *Création d'intrication par une interaction magn´tique*

L'état final est (en utilisant que $|\uparrow\rangle, |\downarrow\rangle$ sont des vecteurs propres de σ_z avec valeurs propres +1 et −1).

$$\begin{aligned} e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\ &= \frac{1}{2}\left(e^{-itJ}|\uparrow\uparrow\rangle - e^{itJ}|\uparrow\downarrow\rangle + e^{itJ}|\downarrow\uparrow\rangle - e^{-itJ}|\downarrow\downarrow\rangle\right) \\ &= \frac{e^{-itJ}}{2}\left(|\uparrow\uparrow\rangle - e^{2itJ}|\uparrow\downarrow\rangle + e^{2itJ}|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right). \end{aligned}$$

1. Pour $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$
   $\Rightarrow |\psi_\tau\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$.

2. Supposons que l'état puisse s'écrire
   $(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle)\otimes(\gamma\,|\uparrow\rangle + \delta\,|\downarrow\rangle) = \alpha\gamma\,|\uparrow\uparrow\rangle + \alpha\delta\,|\uparrow\downarrow\rangle + \beta\gamma\,|\downarrow\uparrow\rangle + \beta\delta\,|\downarrow\downarrow\rangle$,
   alors $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.
   On peut toujours poser $\alpha = 1$ (phase globale). Donc $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i \Rightarrow$ contradiction sur δ. Vous pouvez aussi prendre n'importe quelle valeur fixée pour α pour montrer qu'une contradiction apparait.

Complement on the Hamiltonian in matrix and Dirac notation.

1. **Matrix notation.** In the canonical bases, we have $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Using the tensor product rule one obtains that

$$\begin{aligned} \sigma_1^z\otimes\sigma_2^z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \end{aligned}$$

   thus the Hamiltonian is

$$\mathcal{H} = \begin{pmatrix}\hbar J & 0 & 0 & 0\\ 0 & -\hbar J & 0 & 0\\ 0 & 0 & -\hbar J & 0\\ 0 & 0 & 0 & \hbar J\end{pmatrix}.$$

2. **Dirac notation.** In the bra-ket formalism one has $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, thus

$$\begin{aligned} \sigma_1^z\otimes\sigma_2^z &= (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)\otimes(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ &= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|. \end{aligned}$$

   Therefore, we have

$$\mathcal{H} = \hbar J(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|).$$

3. **Connection between matrix and Dirac notations.** Notice that to verify this one can use

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},$$

which implies that

$$(|\uparrow\rangle\langle\uparrow|)\otimes(|\uparrow\rangle\langle\uparrow|) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}.$$

Similarly one can show that

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix},$$

$$|\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix},$$

$$|\downarrow\downarrow\rangle\langle\downarrow\downarrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}.$$

4. **Eigenvalues and eigenvectors.** One can see that the eigen-values are $\hbar J$ corresponding to the eigen-vectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and $-\hbar J$ corresponding to the eigen-vectors $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$.

Exercice 2 *Rotations on the Bloch sphere*

A general vector can be written in the form $\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

1. The eigenvectors for σ_z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$ which correspond to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$ respectively.

   The eigenvectors for σ_y are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$ which correspond to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$.

   The eigenvectors for σ_x basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$ corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$ respectively.

   The corresponding representation over the Bloch sphere is shown in Figure 1.

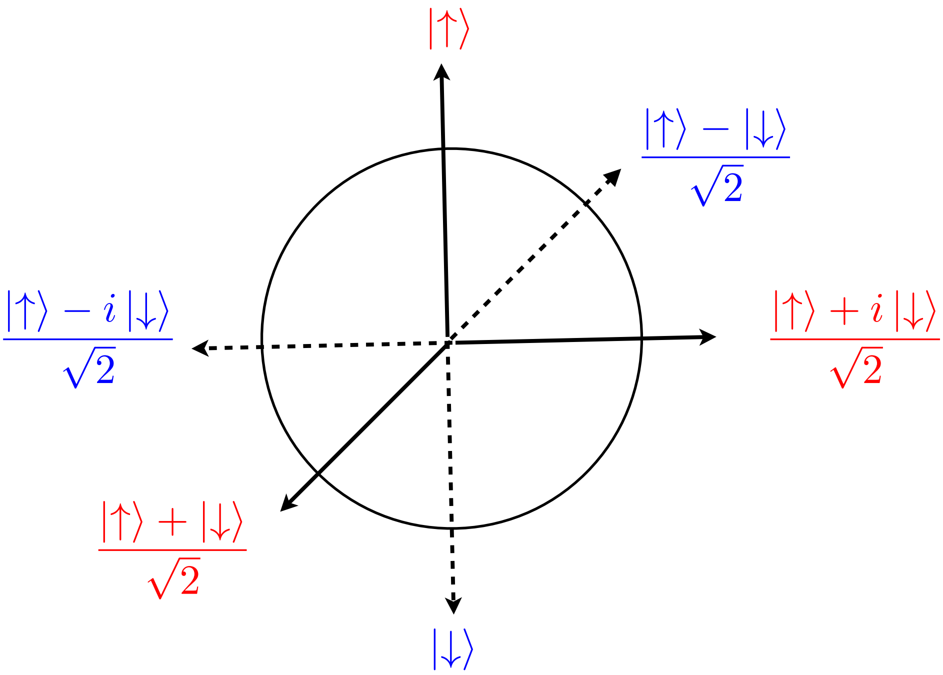

FIGURE 1 – Representation of basis vectors on Bloch Sphere

2. Using the general formula

$$\exp(i\frac{\theta}{2}(\vec{\sigma}.\vec{n})) = \cos(\frac{\theta}{2})I + i\vec{\sigma}.\vec{n}(\sin(\frac{\theta}{2})),$$

   we obtain that

$$\begin{aligned}\exp(-i\frac{\alpha}{2}\sigma_x) &= \cos(\frac{\alpha}{2})I - i\sigma_x(\sin(\frac{\alpha}{2})) \\ &= \begin{pmatrix} \cos(\frac{\alpha}{2}) & -i\sin(\frac{\alpha}{2}) \\ -i\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(-i\frac{\beta}{2}\sigma_y) &= \cos(\frac{\beta}{2})I - i\sigma_y(\sin(\frac{\beta}{2})) \\ &= \begin{pmatrix} \cos(\frac{\beta}{2}) & -\sin(\frac{\beta}{2}) \\ \sin(\frac{\beta}{2}) & \cos(\frac{\beta}{2}) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(-i\frac{\gamma}{2}\sigma_z) &= \cos(\frac{\gamma}{2})I - i\sigma_z(\sin(\frac{\gamma}{2}))\\ &= \begin{pmatrix}\cos(\frac{\gamma}{2}) - i\sin(\frac{\gamma}{2}) & 0\\ 0 & \cos(\frac{\gamma}{2}) + i\sin(\frac{\gamma}{2})\end{pmatrix}\\ &= \begin{pmatrix}e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}.\end{aligned}$$

3. The matrix $\exp(-i\frac{\alpha}{2}\sigma_x)$ is a rotation matrix of angle α around the X-axis, thus the state vector $\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle$ will transform to the vector $\cos(\frac{\theta+\alpha}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})\left|\downarrow\right\rangle$. One can see that geometrically over the Bloch sphere, however one can also show by direct calculation that

$$\exp(-i\frac{\alpha}{2}\sigma_x)\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\right)\left|\downarrow\right\rangle = \cos(\frac{\theta+\alpha}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta+\alpha}{2})\left|\downarrow\right\rangle.$$

Similarly, one can see that $\exp(i\frac{\gamma}{2}\sigma_z)$ is a rotation of angle γ around the Z-axis. Therefore,

$$\exp(-i\frac{\gamma}{2}\sigma_z)\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\theta}{2})\right)\left|\downarrow\right\rangle = e^{-i\frac{\gamma}{2}}\left(\cos(\frac{\theta}{2})\left|\uparrow\right\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin(\frac{\theta}{2})\left|\downarrow\right\rangle\right).$$