ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 18 Principles of Digital Communications
Midterm Exam Apr. 22, 2016

- 4 problems, 35 points, 165 minutes
- This is a closed book exam.
- Only one single-sided handwritten A4 page of summary allowed.

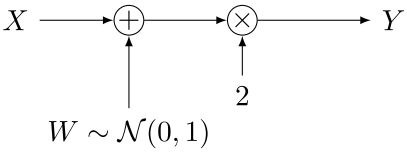
Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS

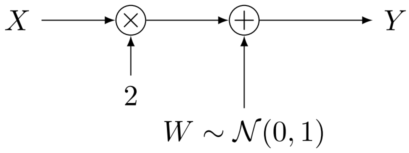
PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET

WE WILL RETURN THE EXAM BACK TO YOU ON WEDNESDAY 27TH OF APRIL IN CLASS. PLEASE VERIFY THE CORRECTIONS RIGHT AWAY. IN CASE YOU HAVE QUESTIONS, ASK US IN CLASS. ONCE YOU TAKE THE EXAM OUT OF THE CLASS ROOM, WE WILL ASSUME THAT YOU AGREE WITH THE CORRECTIONS.

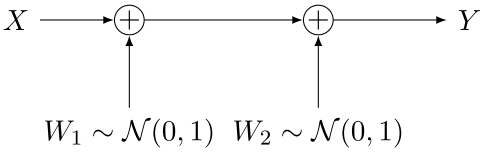
Problem 1. (7 points) Each of the following five channel models with scalar input $X \in \{\pm 1\}$ is "equivalent" to a standard additive Gaussian noise channel with input X and output $Y = X + Z$, where $Z \sim \mathcal{N}(0, \sigma^2)$ for some σ^2. Here, equivalent means that the performance (error probability) of the two systems is identical. For each of the five models below determine the appropriate σ^2 and explain why the two systems are equivalent.



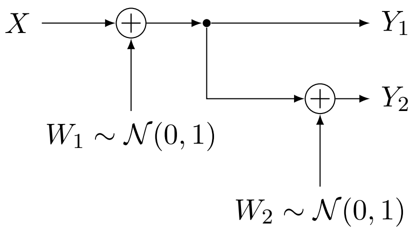
(a) (1 pts)



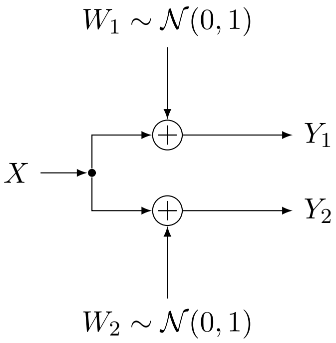
(b) (1 pts)



(c) (1 pts)

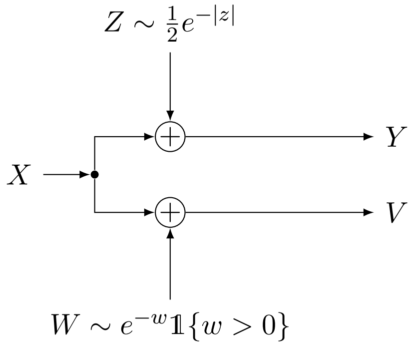


(d) (2 pts)



(e) (2 pts)

Problem 2. (10 points) Consider a communication system where the transmitter uses repetition coding to increase the reliability. That is, to communicate a bit ± 1, it is transmitted via n consecutive independent uses of the communication channel. We also assume $+1$ is communicated with probability p. The receiver observes the transmitter's output via two independent channels as depicted below:



Namely, Y is the transmitted bit corrupted with additive Laplacian noise and V is the transmitted bit contaminated with an Exponentially distributed noise.

(a) (2 pts) Assume the receiver is only allowed to use the output of Laplacian noise channel, which we denote by $(Y_1, \dots, Y_n)$. Derive the decision rule that minimizes the error probability. Show that the decision rule has the form

$$T(y_1, \dots, y_n) \gtreqless \theta(p),$$

where $\theta(p)$ is a threshold that only depends on the prior p and $T(y_1, \dots, y_n)$ is a function that only involves additions and comparisons (so is easy to implement).

(b) (2 pts) Find non-trivial conditions on p under which the decision of the receiver of part (a) is independent of the observation.

(c) (2 pts) Is $T(y_1, \dots, y_n)$ a sufficient statistic for the decision problem of (a)? Justify your answer.

(d) (3 pts) Now suppose the receiver can use outputs of both channels. Namely observes two sequences $(Y_1, \dots, Y_n)$ (which is the output of the Laplacian noise channel) and $(V_1, \dots, V_n)$ (which is the output of the channel with Exponential noise). Note that the observation is now a $2n$-dimensional vector. Show that the hypothesis testing problem has a 2-dimensional sufficient statistic in the form of

$$\left(T(y_1, \dots, y_n), T'(v_1, \dots, v_n)\right),$$

which is computable exclusively using additions and comparisons; derive the optimal decision rule, and sketch the decision regions on (T, T')-plane.

(e) (1 pts) Can a receiver that only observes $(V_1, \dots, V_n)$ perform as good as the optimal receiver you derived in part (d)?

PROBLEM 3. (8 points) The received signal in a communication system is given by

$$R(t) = \begin{cases} w_0(t) + N(t) & \text{if 0 is sent,} \\ w_1(t) + N(t) & \text{if 1 is sent,} \end{cases}$$

where $N(t)$ is white Gaussian noise of spectral density $\frac{N_0}{2}$. $w_0(t)$ and $w_1(t)$ are two different signals of energy $\mathcal{E}$ (i.e., $\|w_0\|^2 = \|w_1\|^2 = \mathcal{E}$) which may or may not be orthogonal. We assume that 0 and 1 are equiprobable.

(a) (2 pts) Let

$$v_0(t) = \frac{w_0(t) - w_1(t)}{\|w_0 - w_1\|}.$$

Find a signal $v_1(t)$ so that $\{v_0(t), v_1(t)\}$ is an orthonormal basis for $\{w_0(t), w_1(t)\}$. Deduce that (U_0, U_1) is a sufficient statistic for the hypothesis testing problem where $U_0 = \langle R, v_0 \rangle$ and $U_1 = \langle R, v_1 \rangle$.

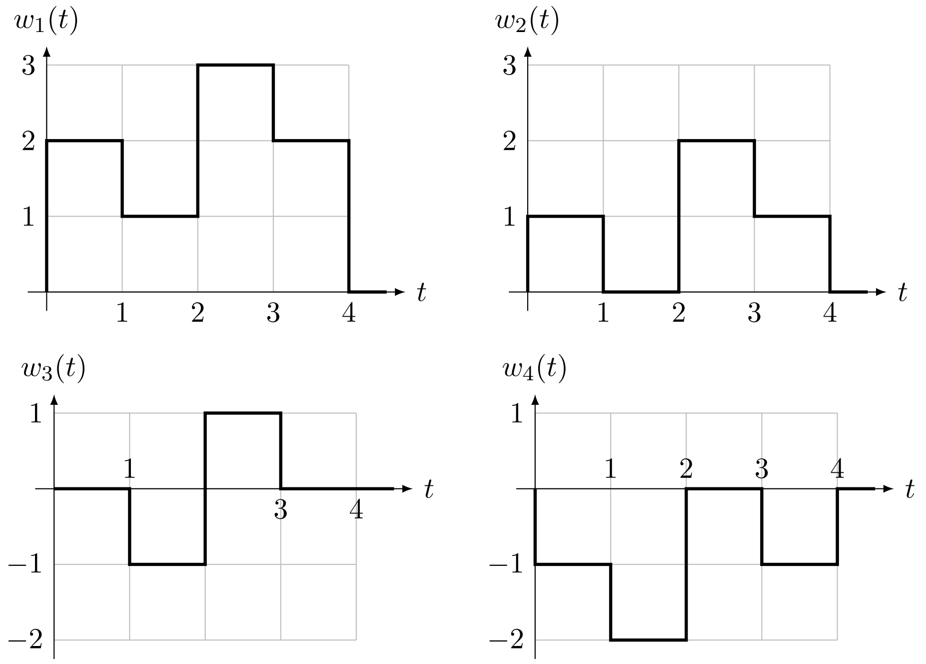
(b) (2 pts) Show that U_0 is a sufficient statistic for the hypothesis testing problem (i.e., U_1 can be thrown away).

(c) (2 pts) Show that the error probability of the MAP decoder is equal to

$$P_e = Q\left(\frac{\|w_0 - w_1\|}{\sqrt{2N_0}}\right).$$

(d) (2 pts) Use the Cauchy–Schwarz inequality to show that $\|w_0 - w_1\| \leq 2\sqrt{\mathcal{E}}$. Deduce that P_e is minimized when $w_1 = -w_0$.

Problem 4. (10 points) In a communication system one of the four signals shown below is chosen and transmitted over an additive white Gaussian noise channel with power spectral density $N_0/2$. Each signal is equally likely to be chosen.



(a) (2 pts) Describe the optimal receiver that decides which signal was transmitted. The n-tuple former must contain a single matched filter with impulse response

$$h(t) = \mathbb{1}\{0 \leq t \leq 1\}.$$

(b) (2 pts) Using the union bound, given an upper bound on the probability of error of the receiver in (a).

(c) (2 pts) Transform the signal set $\{w_1(t), w_2(t), w_3(t), w_4(t)\}$ to a minimum energy signal set. Sketch the new signal set $\{\tilde{w}_1(t), \tilde{w}_2(t), \tilde{w}_3(t), \tilde{w}_4(t)\}$.

(d) (2 pts) If the signal set you found in (c) is used for communication, what will be the probability of error of the optimal receiver?

(e) (2 pts) What is the exact probability of error of the receiver you described in (a)?