Handout 19 Principles of Digital Communications
Solutions to Midterm Exam Apr. 22, 2016

SOLUTION 1.

(a) $\sigma_a^2 = 1$; any invertible transform on the output — in particular multiplication by 2 — does not change the error probability.

(b) $\sigma_b^2 = \frac{1}{4}$; $Y = (\pm 2) + W$ which is equivalent to $Y' = \frac{1}{2}Y = (\pm 1) + \frac{1}{2}W$ and $Z = \frac{1}{2}W \sim \mathcal{N}(0, \frac{1}{4})$.

(c) $\sigma_c^2 = 2$; $Y = (\pm 1) + W_1 + W_2$ and $W_1 + W_2 \sim \mathcal{N}(0, 2)$ (since W_1 and W_2 are independent).

(d) $\sigma_d^2 = 1$; Y_1 is a sufficient statistic for decision.

(e) $\sigma_e^2 = \frac{1}{2}$; the observable is

$$(Y_1, Y_2) = \pm(1, 1) + (W_1, W_2)$$

where $(W_1, W_2) \sim \mathcal{N}(0, I_2)$ and $\frac{1}{2}(Y_1 + Y_2) = \pm 1 + Z$ where $Z = \frac{1}{2}(W_1 + W_2) \sim \mathcal{N}(0, \frac{1}{2})$ is a sufficient statistic for the decision.

SOLUTION 2.

(a) Under the hypothesis $H = +1$, $(Y_1, \ldots, Y_n)$ is an i.i.d. sequence whose components are Laplacian random variables with mean $+1$, namely

$$f_{Y_1,\ldots,Y_n|H}(y_1, \ldots, y_n| + 1) = \left(\frac{1}{2}\right)^n \exp\left\{-\sum_{k=1}^{n} |y_k - 1|\right\}.$$

Similarly,

$$f_{Y_1,\ldots,Y_n|H}(y_1, \ldots, y_n| - 1) = \left(\frac{1}{2}\right)^n \exp\left\{-\sum_{k=1}^{n} |y_k + 1|\right\}.$$

The MAP decision rule is

$$\frac{f_{Y_1,\ldots,Y_n|H}(y_1, \ldots, y_n| + 1)}{f_{Y_1,\ldots,Y_n|H}(y_1, \ldots, y_n| - 1)} \underset{\hat{H}=-1}{\overset{\hat{H}=+1}{\gtrless}} \frac{1-p}{p},$$

which, after canceling the common factors and taking the logarithm, becomes

$$\sum_{k=1}^{n} (|y_k + 1| - |y_k - 1|) \underset{\hat{H}=-1}{\overset{\hat{H}=+1}{\gtrless}} \ln \frac{1-p}{p}. \tag{1}$$

(b) Since $\forall \alpha \in \mathbb{R}$: $|\alpha + 1| - |\alpha - 1| \in [-2, 2]$, the left-hand-side of (1) lies in $[-2n, 2n]$. Therefore, if

$$2n < \ln \frac{1-p}{p} \qquad \iff \qquad p < \frac{1}{1 + e^{2n}},$$

the receiver always chooses $\hat{H} = -1$.

Similarly, if

$$-2n > \ln \frac{1-p}{p} \qquad \iff \qquad p > \frac{e^{2n}}{1 + e^{2n}},$$

the decision will always be $\hat{H} = +1$ (regardless of the observation).

(c) $T(y_1,\ldots,y_n) = \sum_{k=1}^n (|y_k+1| - |y_k-1|)$ is the log-likelihood ratio and, hence, is a sufficient statistic. We can prove this using Neyman–Fisher factorization theorem by noting that (for $a \in \{-1,+1\}$),

$$f_{Y|H}(y_1,\ldots,y_n|a) = \underbrace{\left(\frac{1}{2}\right)^n \exp\left\{-\frac{1}{2}\sum_{k=1}^n (|y_k-1| + |y_k+1|)\right\}}_{h(y_1,\ldots,y_n)} \times \underbrace{\exp\left\{-\frac{a}{2}\sum_{k=1}^n (|y_k-1| - |y_k+1|)\right\}}_{g_a(T(y_1,\ldots,y_n))}. \quad (2)$$

(d) We have

$$f_{V_1,\ldots,V_n|H}(v_1,\ldots,v_n|+1) = \exp\left\{-\sum_{k=1}^n (v_k-1)\right\}\prod_{k=1}^n \mathbb{1}\{v_k \geq 1\},$$

and

$$f_{V_1,\ldots,V_n|H}(v_1,\ldots,v_n|-1) = \exp\left\{-\sum_{k=1}^n (v_k+1)\right\}\prod_{k=1}^n \mathbb{1}\{v_k \geq -1\},$$

Simplifying the above we get (for $a \in \{-1,+1\}$),

$$f_{V_1,\ldots,V_n|H}(v_1,\ldots,v_n|a) = \underbrace{\exp\left\{-\sum_{k=1}^n v_k\right\}}_{h'(v_1,\ldots,v_n)} \times \underbrace{\exp(na)\mathbb{1}\{\min\{v_1,\ldots,v_n\} \geq a\}}_{g'_a(T'(v_1,\ldots,v_n))}, \quad (3)$$

with $T'(v_1,\ldots,v_n) = \min\{v_1,\ldots,v_n\}$.

Since conditioned on $H=a$, $a \in \{-1,+1\}$ the observables $Y_1,\ldots,Y_n$ and $V_1,\ldots,V_n$ are independent,

$$\begin{aligned} & f_{Y_1,\ldots,Y_n,V_1,\ldots,V_n|H}(y_1,\ldots,y_n,v_1,\ldots,y_n|a) \\ &= f_{Y_1,\ldots,Y_n|H}(y_1,\ldots,y_n|a) \times f_{V_1,\ldots,V_n|H}(v_1,\ldots,y_n|a) \\ &= h(y_1,\ldots,y_n)h'(v_1,\ldots,v_n) \times g_a\big(T(y_1,\ldots,y_n)\big)g'_a\big(T'(v_1,\ldots,v_n)\big) \end{aligned}$$

where h, g_a, h', and g'_a are defined in (2) and (3). Therefore, using the factorization theorem we conclude that $\big(T(y_1,\ldots,y_n), T'(v_1,\ldots,v_n)\big)$ is a sufficient statistic for the hypothesis testing problem.

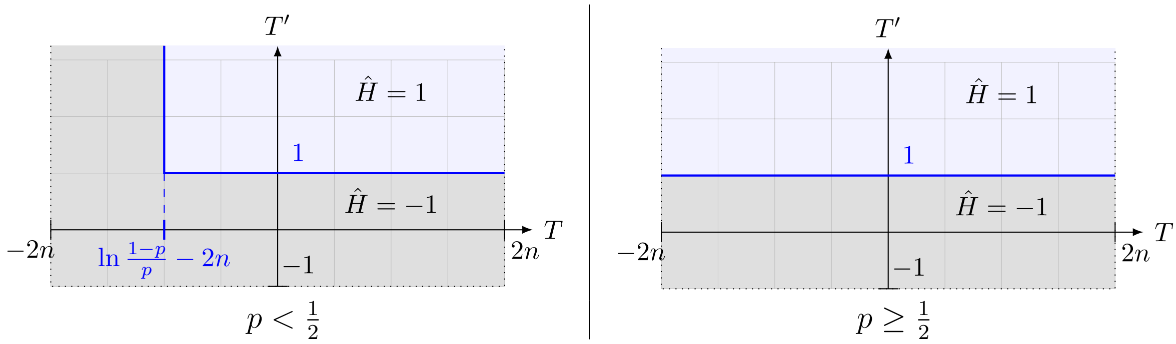
The MAP decision rule (in terms of T and T') is

$$g_{+1}(T)g'_{+1}(T') \times p \underset{\hat{H}=-1}{\overset{\hat{H}=+1}{\gtrless}} g_{-1}(T)g'_{-1}(T') \times (1-p). \quad (4)$$

Now if $T' = \min\{v_1,\ldots,v_n\} \in (-1,1)$ we see that $g'_{+1}(T') = 0$ thus the MAP rule always chooses $\hat{H} = -1$. Otherwise (i.e., when $\min\{v_1,\ldots,v_n\} \geq 1$) (4) reduces to

$$T(y_1,\ldots,y_n) = \sum_{k=1}^n (|y_k+1| - |y_k-1|) \underset{\hat{H}=-1}{\overset{\hat{H}=+1}{\gtrless}} \ln\frac{1-p}{p} - 2n. \quad (5)$$

Thus, the decision regions are:



(note that $T \in [-2n, 2n]$ as we discussed in (b) and $T' \geq -1$).

(e) From the decision regions of (d) it is clear that if $p \geq \frac{1}{2}$ the optimal decision depends only on T' which, in turn, is only a function of $(V_1, \dots, V_n)$. Therefore, if $p \geq \frac{1}{2}$ the receiver that only observes $(V_1, \dots, V_n)$ can perform as well as the optimal receiver.

Solution 3.

(a) Since the space spanned by $\{w_0, w_1\}$ is the same as the space spanned by $\{v_0, w_1\}$, we can obtain v_1 by applying the Gram–Schmidt procedure on $\{v_0, w_1\}$:

$$
\begin{aligned}
w_1 - \langle w_1, v_0 \rangle v_0 &= w_1 - \left\langle w_1, \frac{w_0 - w_1}{\|w_0 - w_1\|} \right\rangle \frac{w_0 - w_1}{\|w_0 - w_1\|} \\
&= w_1 - \frac{\langle w_0, w_1 \rangle - \|w_1\|^2}{\|w_0 - w_1\|^2} \cdot (w_0 - w_1) \\
&= w_1 - \frac{\langle w_0, w_1 \rangle - \|w_1\|^2}{\|w_0\|^2 + \|w_1\|^2 - 2\langle w_0, w_1 \rangle} \cdot (w_0 - w_1) \\
&= w_1 - \frac{\langle w_0, w_1 \rangle - \mathcal{E}}{2\mathcal{E} - 2\langle w_0, w_1 \rangle} \cdot (w_0 - w_1) \\
&= w_1 + \frac{1}{2}(w_0 - w_1) = \frac{1}{2} \cdot (w_0 + w_1).
\end{aligned}
$$

Therefore,

$$
v_1 = \frac{w_1 - \langle w_1, v_0 \rangle v_0}{\|w_1 - \langle w_1, v_0 \rangle v_0\|} = \frac{w_0 + w_1}{\|w_0 + w_1\|}.
$$

(b) Let $Z_0 = \langle N, v_0 \rangle$ and $Z_1 = \langle N, v_1 \rangle$. Z_0 and Z_1 are independent because v_0 and v_1 are orthogonal. We have:

$$
\begin{aligned}
U_1 = \langle R, v_1 \rangle &= \begin{cases} \langle w_0, \frac{w_0 + w_1}{\|w_0 + w_1\|} \rangle + Z_1 & \text{if 0 is sent,} \\ \langle w_1, \frac{w_0 + w_1}{\|w_0 + w_1\|} \rangle + Z_1 & \text{if 1 is sent.} \end{cases} \\
&= \begin{cases} \frac{\|w_0\|^2 + \langle w_0, w_1 \rangle}{\|w_0 + w_1\|} + Z_1 & \text{if 0 is sent,} \\ \frac{\langle w_1, w_0 \rangle + \|w_0\|^2}{\|w_0 + w_1\|} + Z_1 & \text{if 1 is sent.} \end{cases} \\
&= \begin{cases} \frac{\mathcal{E} + \langle w_0, w_1 \rangle}{\|w_0 + w_1\|} + Z_1 & \text{if 0 is sent,} \\ \frac{\mathcal{E} + \langle w_0, w_1 \rangle}{\|w_0 + w_1\|} + Z_1 & \text{if 1 is sent.} \end{cases}
\end{aligned}
$$

This shows that the distribution of U_1 is independent from the transmitted bit (and from U_0). Therefore, U_1 can be thrown away. Hence, U_0 is sufficient statistics for the hypothesis testing problem.

(c) We have:

$$U_0 = \langle R, v_0 \rangle = \begin{cases} \langle w_0, \frac{w_0 - w_1}{\|w_0 - w_1\|} \rangle + Z_0 & \text{if 0 is sent,} \\ \langle w_1, \frac{w_0 - w_1}{\|w_0 - w_1\|} \rangle + Z_0 & \text{if 1 is sent.} \end{cases}$$

$$= \begin{cases} \frac{\|w_0\|^2 - \langle w_0, w_1 \rangle}{\|w_0 - w_1\|} + Z_0 & \text{if 0 is sent,} \\ \frac{\langle w_1, w_0 \rangle - \|w_0\|^2}{\|w_0 - w_1\|} + Z_0 & \text{if 1 is sent.} \end{cases}$$

$$= \begin{cases} \frac{\mathcal{E} - \langle w_0, w_1 \rangle}{\|w_0 - w_1\|} + Z_0 & \text{if 0 is sent,} \\ \frac{\langle w_0, w_1 \rangle - \mathcal{E}}{\|w_0 - w_1\|} + Z_0 & \text{if 1 is sent.} \end{cases}$$

Note that $\|w_0 - w_1\|^2 = \|w_0\|^2 + \|w_1\|^2 - 2\langle w_0, w_1 \rangle = 2\mathcal{E} - 2\langle w_0, w_1 \rangle$. Therefore,

$$U_0 = \begin{cases} \frac{\|w_0 - w_1\|^2}{2\|w_0 - w_1\|} + Z_0 & \text{if 0 is sent,} \\ \frac{-\|w_0 - w_1\|^2}{2\|w_0 - w_1\|} + Z_0 & \text{if 1 is sent.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\|w_0 - w_1\| + Z_0 & \text{if 0 is sent,} \\ -\frac{1}{2}\|w_0 - w_1\| + Z_0 & \text{if 1 is sent.} \end{cases}$$

Now since $Z_0 = \langle N, v_0 \rangle \sim \mathcal{N}(0, \frac{N_0}{2})$, the probability of error of the MAP decoder is given by

$$P_e = Q\left(\frac{\frac{1}{2}\|w_0 - w_1\|}{\sqrt{\frac{N_0}{2}}} \right) = Q\left(\frac{\|w_0 - w_1\|}{\sqrt{2N_0}} \right).$$

(d) The Cauchy–Schwarz inequality gives $|\langle w_0, w_1 \rangle| \leq \|w_0\| \cdot \|w_1\| = \mathcal{E}$. Therefore, $\langle w_0, w_1 \rangle \geq -\mathcal{E}$. Hence,

$$\|w_0 - w_1\|^2 = 2\mathcal{E} - 2\langle w_0, w_1 \rangle \leq 2\mathcal{E} + 2\mathcal{E} = 4\mathcal{E}.$$

We conclude that $\|w_0 - w_1\| \leq 2\sqrt{\mathcal{E}}$. Therefore, the probability of error of the MAP decoder is lower-bounded as follows:

$$P_e = Q\left(\frac{\|w_0 - w_1\|}{\sqrt{2N_0}} \right) \overset{(\star)}{\geq} Q\left(\frac{2\sqrt{\mathcal{E}}}{\sqrt{2N_0}} \right) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \right).$$

Moreover, $(\star)$ becomes an equality when $\langle w_0, w_1 \rangle = -\mathcal{E} = -\|w_0\| \cdot \|w_1\|$, which is true if $w_1 = -w_0$.

Solution 4.

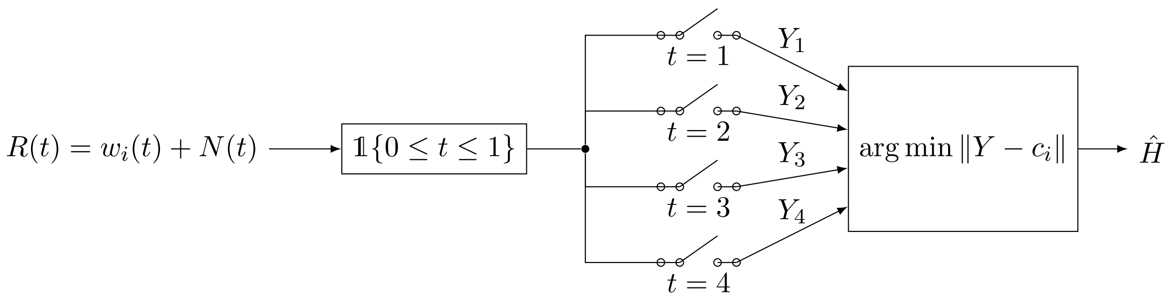
(a) Looking at the waveforms we realize that the four signals $\psi_1(t) = \mathbb{1}\{0 \leq t \leq 1\}$, $\psi_2(t) = \psi_1(t-1)$, $\psi_3(t) = \psi_1(t-2)$, and $\psi_3(t) = \psi_1(t-2)$ form an orthonormal basis for the signal space spanned by the waveforms. In this basis $w_1(t)$, $w_2(t)$, $w_3(t)$, and $w_4(t)$ correspond to the codewords $c_1 = (2, 1, 3, 2)$, $c_2 = (1, 0, 2, 1)$, $c_3 = (0, -1, 1, 0)$ and $c_4 = (-1, -2, 0, -1)$ respectively.

An ML receiver (which is optimal because of equiprobable hypotheses) first projects the received signal $R(t) = w_i(t) + N(t)$ onto the orthonormal basis and forms the 4-tuple (Y_1, Y_2, Y_3, Y_4) with $Y_k = \langle R(t), \psi_k(t) \rangle$, $k = 1, 2, 3, 4$. This reduces the problem to the hypothesis testing problem in discrete additive white Gaussian noise channel,

$$\text{under } H = i,\ i = 1, 2, 3, 4: \qquad Y = c_i + Z$$

where c_i's are defined above and $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_4)$. We know that the ML receiver should chose $\hat{H} = \arg\min_i \|Y - c_i\|$.

We finally realize that since $h(t) = \psi_1(1-t)$ and the remaining basis vectors are the shifted versions of $\psi_1(t)$, the n-tuple former can be implemented by sampling the output of a single filter at times $t = 1, 2, 3$ and 4 to compute Y_1, Y_2, Y_3, and Y_4 respectively:



(b) The union bound gives

$$\Pr\{\text{error}|w_i \text{ is sent}\} \leq \sum_{j \neq i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

where $d_{i,j} = \|w_i - w_j\| = \|c_i - c_j\|$. In the following table we have computed those values

$d_{i,j}$	1	2	3	4
1	0	2	4	6
2	2	0	2	4
3	4	2	0	2
4	6	4	2	0

Consequently,

$$\Pr\{\text{error}|w_1 \text{ is sent}\} = \Pr\{\text{error}|w_4 \text{ is sent}\} = Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + Q\left(\frac{6}{\sqrt{2N_0}}\right),$$

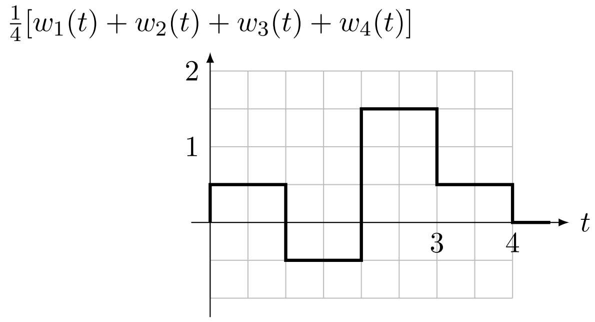
and

$$\Pr\{\text{error}|w_2 \text{ is sent}\} = \Pr\{\text{error}|w_3 \text{ is sent}\} = 2Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right).$$

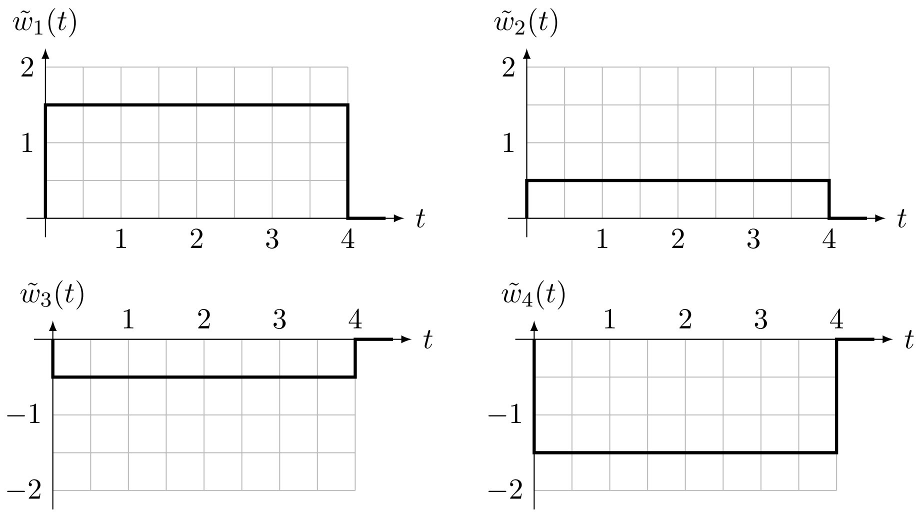
Therefore,

$$\begin{aligned}\Pr\{\text{error}\} &= \sum_{i=1}^{4} \Pr\{w_i \text{ is sent}\} \Pr\{\text{error}|w_i \text{ is sent}\} \\ &\leq \frac{3}{2} Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + \frac{1}{2} Q\left(\frac{6}{\sqrt{2N_0}}\right).\end{aligned}$$

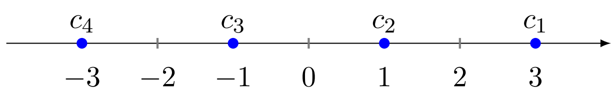
(c) The minimum energy signal set is obtained by subtracting from each signal the average $\frac{1}{4}[w_1(t) + w_2(t) + w_3(t) + w_4(t)]$ which is depicted below



Therefore the minimum energy signal set is



(d) It is easy to verify that the new signal set spans a one-dimensional space with basis $\tilde{\psi}(t) = \frac{1}{2}\mathbb{1}\{0 \le t \le 4\}$. Indeed, the new signal set corresponds to 4-PAM constellation



For the 4-PAM constellation,

$$\Pr\{\text{error}|w_1 \text{ is sent}\} = \Pr\{\text{error}|w_4 \text{ is sent}\} = Q\left(\frac{2}{\sqrt{2N_0}}\right),$$

and

$$\Pr\{\text{error}|w_2 \text{ is sent}\} = \Pr\{\text{error}|w_3 \text{ is sent}\} = 2Q\left(\frac{2}{\sqrt{2N_0}}\right),$$

which yields

$$\Pr\{\text{error}\} = \frac{3}{2}Q\left(\frac{2}{\sqrt{2N_0}}\right).$$

(e) Since translation is an isometric transform and does not change the probability of error, the probability of error for the receiver in part (a) will also be equal to $\frac{3}{2}Q\left(\frac{2}{\sqrt{2N_0}}\right)$.