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School of Computer and Communication Sciences

Handout 15 Principles of Digital Communications
Solutions to Problem Set 7 Apr. 19, 2016

Solution 1.

(a) The Cauchy–Schwarz inequality states

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

with equality if and only if $x = \alpha y$ for some scalar α. For our problem, we can write

$$|\langle w, \phi\rangle|^2 \leq \|w\|^2 \cdot \|\phi\|^2 = \|w\|^2$$

with equality if and only if $\phi = \alpha w$ for some scalar α. Thus, the maximizing $\phi(t)$ is simply a scaled version of $w(t)$.

Remark. *In two dimensions, we have $|\langle x, y\rangle| = \|x\| \cdot \|y\| \cos\alpha$, where α is the angle between the two vectors. It is clear that the maximum is achieved when $\cos\alpha = 1 \Leftrightarrow \alpha = 0$ (or $\alpha = k2\pi$). Thus, x and y are colinear.*

(b) The problem is

$$\max_{\phi_1,\phi_2} (c_1\phi_1 + c_2\phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1$$

Thus, we can reduce by setting $\phi_2 = \sqrt{1-\phi_1^2}$ to obtain

$$\max_{\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right)$$

This maximum is found by taking the derivative:

$$\frac{d}{d\phi_1}\left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right) = c_1 - c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$$

Setting this equal to zero yields $c_1 = c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$, i.e,

$$c_1^2 = c_2^2\frac{\phi_1^2}{1-\phi_1^2}$$

This immediately gives $\phi_1 = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and thus $\phi_2 = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$, which are colinear to c_1 and c_2 respectively.

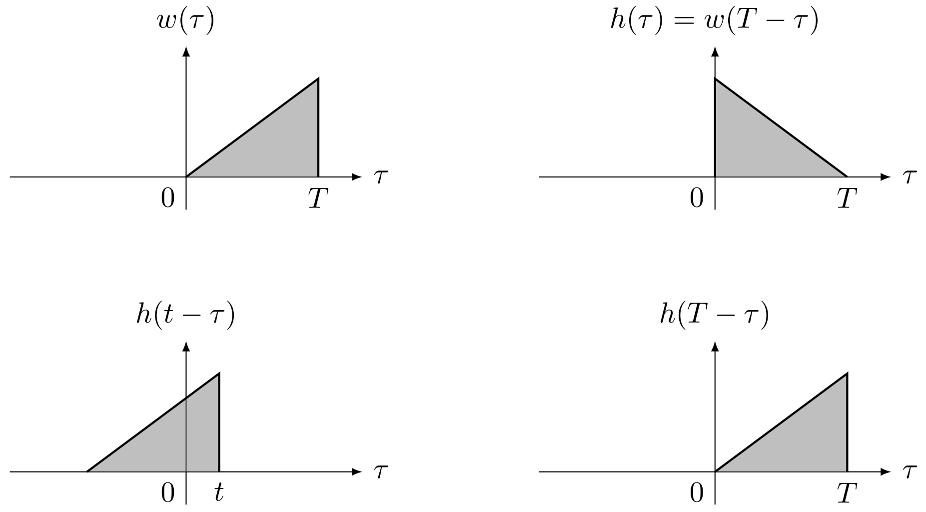
Note: the goal of this exercise was to display yet another way to derive the matched filter.

(c) Passing an input $w(t)$ through a filter with impulse response $h(t)$ generates output waveform $y(t) = \int w(\tau)h(t-\tau)d\tau$. If this waveform $y(t)$ is sampled at time $t = T$, then the output sample is

$$y(T) = \int w(\tau)h(T-\tau)d\tau \tag{1}$$

An example signal $w(\tau)$ is shown below (top left). The filter is then the waveform shown on the top right, and the convolution term of the filter on the bottom left. Finally, the filter term $h(T-\tau)$ of Equation (1) is shown on the bottom right. One can see that $h(T-\tau) = w(\tau)$, so indeed

$$y(T) = \int w(\tau)h(T-\tau)d\tau = \int w^2(\tau)d\tau = \int_0^T w^2(\tau)d\tau$$



SOLUTION 2.

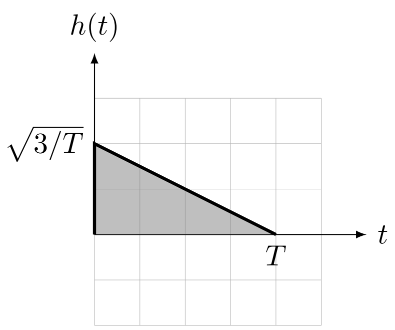
(a) The binary hypothesis testing problem may be written as:

$$\begin{aligned} H = 0: &\quad R(t) = w_1(t) + N(t) \\ H = 1: &\quad R(t) = w_2(t) + N(t) \end{aligned}$$

The impulse response of a matched filter is

$$h(t) = \frac{w_1(T-t)}{\|w_1(t)\|}$$

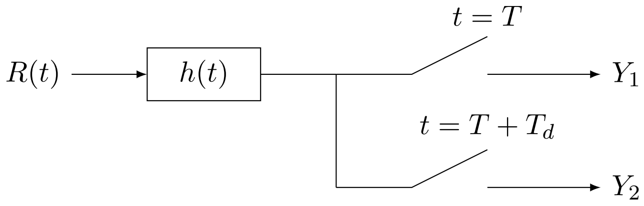
and is shown below. We have normalized the impulse response of the matched filter to have unit norm. Note that this does not affect the probability of error.



The output of the matched filter sampled at $t = T$ and $t = T + T_d$ is $Y_1 = \langle R(t), \frac{w_1(t)}{\|w_1\|}\rangle$ and $Y_2 = \langle R(t), \frac{w_2(t)}{\|w_2\|}\rangle$ respectively. The decision rule is

$$Y_1 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} Y_2$$

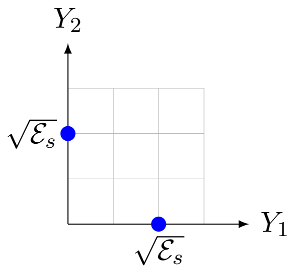
The block diagram of the system is shown below.



(b) For $T_d \geq T$, the signals $w_1(t)$ and $w_2(t)$ are orthogonal to each other. Let

$$\mathcal{E}_s = \|w_1\|^2 = \frac{A^2 T}{3}$$

(The signal space representation of the constellation can be seen below.)



The noise $Z_1, Z_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ and Z_1 is independent of Z_2. The probability of error can be readily calculated as

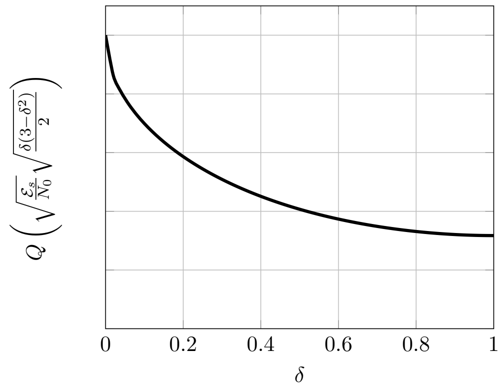
$$P_e = Q\left(\frac{\sqrt{2\mathcal{E}_s}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

For $T_d \leq T$ we note that (since the receiver has not changed) still $(Z_1, Z_2) \sim \mathcal{N}(0, \frac{N_0}{2} I)$ and, hence, the error probability equals $Q(\frac{d}{2\sqrt{N_0/2}})$ where d is the distance between two codewords. Therefore, we compute

$$\begin{aligned}
\|w_1(t) - w_2(t)\|^2 &= \int (w_1(t) - w_2(t))^2 dt \\
&= \int_0^{T_d} \left(\frac{A}{T}\right)^2 t^2 dt + \int_{T_d}^{T} \left(T_d \frac{A}{T}\right)^2 dt + \int_T^{T+T_d} \left(\frac{A}{T}\right)^2 (t - T_d)^2 dt \\
&= \left(\frac{A}{T}\right)^2 \left[\frac{T_d^3}{3} + T_d^2 (T - T_d) + \frac{T^3 - (T - T_d)^3}{3}\right] \\
&\overset{(\star)}{=} \left(\frac{A}{T}\right)^2 \frac{1}{3} T^3 \delta (3 - \delta^2) \\
&= \mathcal{E}_s \delta (3 - \delta^2)
\end{aligned}$$

where in $(\star)$ we have defined $\delta = \frac{T_d}{T}$. Given this, we can compute

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}} \sqrt{\frac{\delta(3 - \delta^2)}{2}}\right)$$

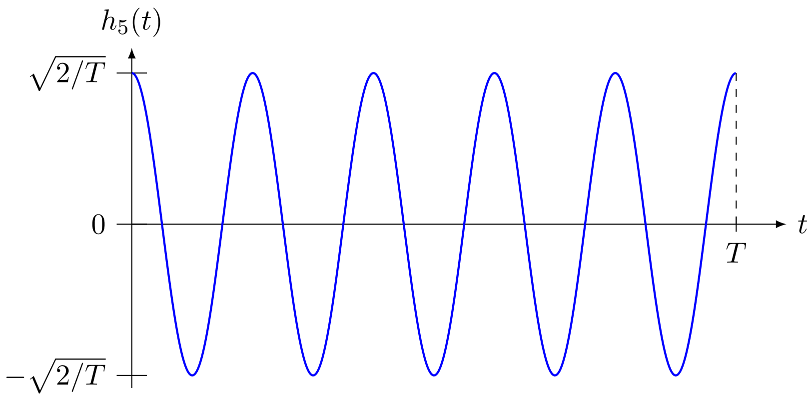


SOLUTION 3.

(a) The matched filter is the filter whose impulse response is a delayed, time-reversed version of $w_j(t)$, i.e.

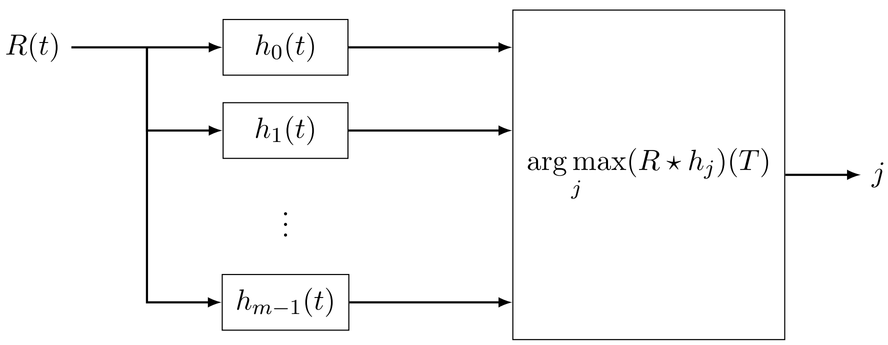
$$\begin{aligned} h_j(t) &= w_j(T-t) = \sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n_j(T-t)}{T}\right)\mathbb{1}_{[0,T]}(T-t) \\ &= \sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n_j t}{T}\right)\mathbb{1}_{[0,T]}(t) \end{aligned}$$

As an example, $h_5(t)$ is shown below.



The receiver then processes the received signal $R(t)$ through the matched filter $h_j(t)$ to obtain $(R \star h_j)(t)$. This signal is sampled at time T to yield the value needed for the MAP decision.

(b) We need m matched filters, one for each signal.

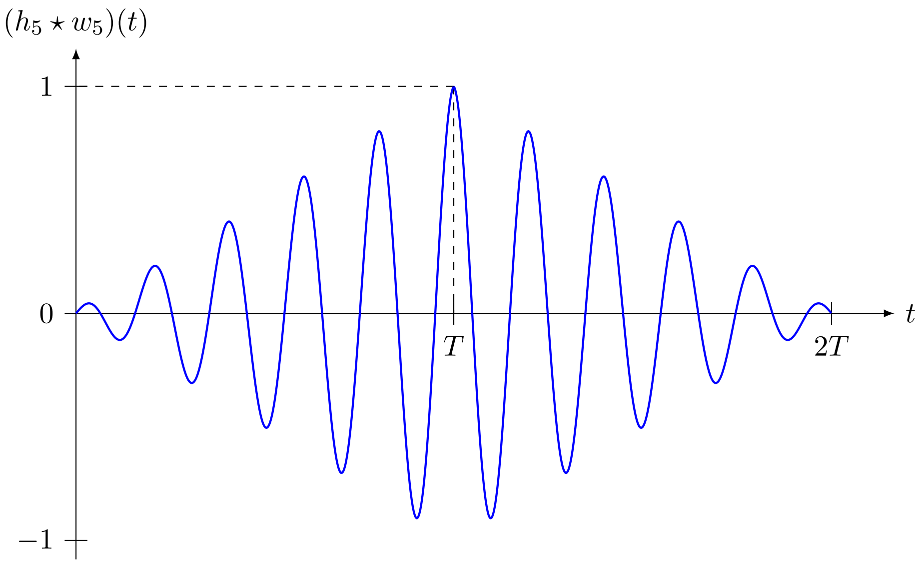


(c) The following **matlab** program computes the output of the matched filter $h_5(t)$.

```
T = 1;
Resolution = 1e-3;
t = 0:Resolution:T;
nj = 5;

wj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );
hj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );

output = conv(wj, hj);
```



Note that the resulting signal is *zero* for $t \leq 0$ and also for $t \geq 2T$. The figure also reveals why sampling at time $t = T$ is a good idea: the value of the matched filter output signal is maximal.

Solution 4. Note that the decision statistic Y is given by

$$Y = \begin{cases} (w \star h)(t_0) + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent,} \end{cases}$$

where $Z = (N \star h)(t_0) \sim \mathcal{N}\left(0, \frac{N_0}{2}\|h\|^2\right)$.

(a) With the given choice of $h(t)$ and t_0, we find $(w \star h)(t_0) = \frac{1}{6}$ and $\|h\|^2 = \frac{4}{3}$, so the decision statistic Y is given by

$$Y = \begin{cases} \frac{1}{6} + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent,} \end{cases}$$

and the ML rule will compare Y to the threshold $\frac{1}{12}$. The resulting error probability is then

$$P_e = Q\left(\frac{1}{12}\frac{1}{\sqrt{\frac{N_0}{2}\|h\|^2}}\right) = Q\left(\frac{1}{12}\sqrt{\frac{3}{2N_0}}\right)$$

(b) Yes. With the choice $t_0 = 4$, we would be implementing the matched filter which we know to be optimal. Indeed the decision statistic will be

$$Y = \begin{cases} \frac{4}{3} + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent,} \end{cases}$$

with Z as defined above. The error probability thus becomes

$$P_e = Q\left(\frac{2}{3}\frac{1}{\sqrt{\frac{N_0}{2}\|h\|^2}}\right) = Q\left(\frac{2}{3}\sqrt{\frac{3}{2N_0}}\right)$$

(c) With this new choice of $h(t)$ and t_0, we find $(w \star h)(t_0) = 1$ and $\|h\|^2 = 2$. The ML rule will then compare Y to the threshold $\frac{1}{2}$ and the resulting error probability will be

$$P_e = Q\left(\frac{1}{2}\frac{1}{\sqrt{\frac{N_0}{2}\|h\|^2}}\right) = Q\left(\frac{1}{2}\sqrt{\frac{1}{N_0}}\right)$$

(d) We know that the matched filter $h_{opt}(t) = w(4-t)$ gives the optimal statistic Y_{opt} when sampled at $t_0 = 4$, but since we only have $h(t) = \mathbb{1}_{[0,2]}(t)$, the best approximation possible for $h_{opt}(t)$ is given by $h_{approx}(t) = h(4-t) - h(2-t)$. Sampling $h_{approx}(t)$ at $t_0 = 4$ to obtain the statistic Y is identical to sampling $h(t)$ at $t_0 = 2$ and $t_1 = 4$ to obtain Y_0 and Y_1 respectively, then form $Y = Y_0 - Y_1$. The decision statistic will be

$$Y = \begin{cases} 2 + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent,} \end{cases}$$

with $Z \sim \mathcal{N}\left(0, \frac{N_0}{2}\|h\|^2 + \frac{N_0}{2}\|h\|^2\right)$. The error probability thus becomes

$$P_e = Q\left(\sqrt{\frac{1}{2N_0}}\right)$$

Solution 5.

(a) The third component of c_i is zero for all i. Furthermore Z_1, Z_2 and Z_3 are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1}) f_{Z_2}(y_2 - c_{i,2}) f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^\mathsf{T}$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher–Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) We have $Y_3 = Z_3 = Z_2$. By observing Y_3, we can remove the noise in the second component of Y. Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible using only $(Y_1, Y_2)^\mathsf{T}$ (see the next question for more on this). We can see that Y_3 contains very useful information and can't be discarded. Therefore, $(Y_1, Y_2)^\mathsf{T}$ is not a sufficient statistic.

(c) If we have only $(Y_1, Y_2)^\mathsf{T}$ then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = \{0, 1\}$$

Using the fact that $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$, the ML test becomes

$$y_1 - y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$, and so the error probability will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

Now assume that we have access to Y_1, Y_2 and Y_3. Y_3 contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver we can observe the second component of c_i without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:

$$\hat{H} = \begin{cases} 0 & y_2 - y_3 = 0 \\ 1 & y_2 - y_3 = 1 \end{cases}$$

Clearly this decision rule minimizes the error probability. This shows once again that $(Y_1, Y_2)^\mathsf{T}$ can't be a sufficient statistic.

Solution 6.

(a) The optimal solution is to pass $R(t)$ through the matched filter $w(T - t)$ and sample the result at $t = T$ to get a sufficient statistic denoted by Y. (In this problem, $T = 1$.) Note that $Y = S + N$, where S and N are random variables denoting the signal and the noise components respectively. Under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $3c$, c, $-c$ and $-3c$ respectively.

Let $\hat{X}$ be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of $\hat{X}$ in the following fashion:

$$\hat{X} = \begin{cases} +3, & Y \in [2c, \infty) \\ +1, & Y \in [0, 2c) \\ -1, & Y \in [-2c, 0) \\ -3, & Y \in [-\infty, -2c) \end{cases} \tag{2}$$

(b) The probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4}\left[Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + Q\left(\frac{c}{\sqrt{N_0/2}}\right)\right] \\ &= \frac{3}{2}Q\left(\frac{c}{\sqrt{N_0/2}}\right) \end{aligned}$$

(c) In this case under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $\frac{9c}{4}$, $\frac{3c}{4}$, $\frac{-3c}{4}$ and $\frac{-9c}{4}$ respectively. Using the decision rule in (2), the probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4} \left[Q\left(\frac{c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}} \right) \right. \\ &\quad \left. + Q\left(\frac{5c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{c/4}{\sqrt{N_0/2}} \right) \right] \\ &= \frac{1}{2} \left[Q\left(\frac{c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}} \right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}} \right) \right] \end{aligned}$$

(d) The noise process $N(t)$ is a stationary Gaussian random process. So the noise component N (which is the sample of match-filter output at time T) is a Gaussian random variable with mean

$$\mathbb{E}[N] = \mathbb{E}\left[\int_{-\infty}^{\infty} N(t) w(t) dt \right] = \mathbb{E}\left[\int_0^1 N(t) dt \right] = 0$$

Because the process $N(t)$ is stationary, without loss of generality we choose the boundaries of the integral to be 0 and T where in this problem $T = 1$.

Now, let us calculate the noise variance.

$$\begin{aligned} \text{var}(N) &= \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \mathbb{E}[N^2] \\ &= \mathbb{E}\left[\int_{-\infty}^{\infty} N(t) w(t) dt \int_{-\infty}^{\infty} N(v) w(v) dv \right] \\ &= \mathbb{E}\left[\int_0^1 N(t) dt \int_0^1 N(v) dv \right] \\ &= \mathbb{E}\left[\int_0^1 \int_0^1 N(t) N(v) dt dv \right] \\ &= \int_0^1 \int_0^1 K_N(t - v) dt dv \\ &= \int_0^1 \int_0^1 \frac{1}{4\alpha} e^{-|t-v|/\alpha} dt dv \\ &= \frac{1}{2} \left(\alpha \left(e^{-1/\alpha} - 1 \right) + 1 \right) \end{aligned}$$

Thus the new probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4} \left[Q\left(\frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}} \right) + 2Q\left(\frac{c}{\sqrt{\text{var}(N)}} \right) + Q\left(\frac{c}{\sqrt{\text{var}(N)}} \right) \right] \\ &= \frac{3}{2} Q\left(\frac{c}{\sqrt{\frac{1}{2} \left(\alpha \left(e^{-1/\alpha} - 1 \right) + 1 \right)}} \right) \end{aligned}$$