ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

**Handout 14** Principles of Digital Communications
Problem Set 7 Apr. 13, 2016

PROBLEM 1. In this problem, we develop further intuition about matched filters. You may assume that all waveforms are real-valued. Let $R(t) = \pm w(t) + N(t)$ be the channel output, where $N(t)$ is additive white Gaussian noise of power spectral density $\frac{N_0}{2}$ and $w(t)$ is an arbitrary but fixed pulse. Let $\phi(t)$ be a unit-norm but otherwise arbitrary pulse, and consider the receiver operation

$$Y = \langle R, \phi \rangle = \langle w, \phi \rangle + \langle N, \phi \rangle$$

The signal-to-noise ratio (SNR) is defined as

$$\text{SNR} \triangleq \frac{|\langle w, \phi \rangle|^2}{\mathbb{E}[|\langle N, \phi \rangle|^2]}$$

Notice that the SNR remains the same if we scale $\phi(t)$ by a constant factor. Notice also that

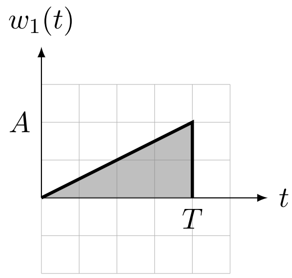
$$\mathbb{E}[|\langle N, \phi \rangle|^2] = \frac{N_0}{2}$$

(a) Use the Cauchy–Schwarz inequality to give an upper bound on the SNR. What is the condition for equality in the Cauchy–Schwarz inequality? Find the $\phi(t)$ that maximizes the SNR. What is the relationship between the maximizing $\phi(t)$ and the signal $w(t)$?

(b) Let us verify that we would get the same result using a pedestrian approach. Instead of waveforms we consider tuples. So let $c = (c_1, c_2)^\mathsf{T} \in \mathbb{R}^2$ and use calculus (instead of the Cauchy–Schwarz inequality) to find the $\phi = (\phi_1, \phi_2)^\mathsf{T} \in \mathbb{R}^2$ that maximizes $\langle c, \phi \rangle$ subject to the constraint that $\phi$ has unit norm.

(c) Verify with a picture (convolution) that the output at time $T$ of a filter with input $w(t)$ and impulse response $h(t) = w(T - t)$ is indeed $\langle w, w \rangle = \int_{-\infty}^{\infty} w^2(t)dt$.

PROBLEM 2. Let $w_1(t)$ be as shown below and let $w_2(t) = w_1(t - T_d)$, where $T_d \geq T$ is a fixed number known to the receiver. One of the two pulses is selected at random and transmitted across the AWGN channel of noise power spectral density $\frac{N_0}{2}$.




(a) Describe an ML receiver that decides which pulse was transmitted. The $n$-tuple former must contain a single causal matched filter. Finally, draw the matched filter impulse response.

(b) Express the error probability of the receiver in (a) in terms of $A, T, T_d, N_0$. Consider both cases $T_d \geq T$ and $T_d < T$.

PROBLEM 3. In this problem, we consider the implementation of matched filter receivers. In particular, we consider frequency-shift keying (FSK) with the following signals:

$$w_j(t) = \begin{cases} \sqrt{\frac{2}{T}} \cos 2\pi \frac{n_j}{T} t & 0 \leq t \leq T \\ 0 & \text{otherwise,} \end{cases}$$

where $n_j \in \mathbb{Z}$ and $0 \leq j \leq m-1$. Thus, the communication scheme consists of $m$ signals $w_j(t)$ of different frequencies $\frac{n_j}{T}$.

(a) Determine the impulse response $h_j(t)$ of a causal matched filter for the signal $w_j(t)$. Plot $h_j(t)$ and specify the sampling time.

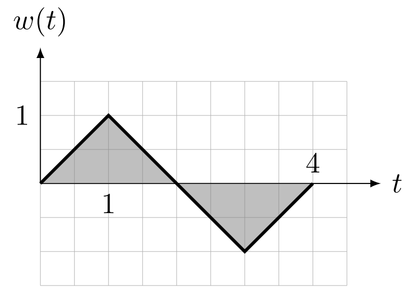
(b) Sketch the matched filter receiver. How many matched filters are needed?

(c) Sketch the output of the matched filter with impulse response $h_j(t)$ when the input is $w_j(t)$.

PROBLEM 4. The received signal $R(t)$ in a communication system is given by

$$R(t) = \begin{cases} w(t) + N(t) & \text{if 1 is sent} \\ N(t) & \text{if 0 is sent,} \end{cases}$$

where $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$ and $w(t)$ is as shown below.




At the receiver, the signal $R(t)$ is passed through a filter with impulse response $h(t)$ and the output of the filter is sampled at time $t_0$ to yield a decision statistic $Y$. A maximum likelihood decision rule is then used based on $Y$ to decide if 1 or 0 was sent.

(a) For $h(t) = w(4-t)$, find the error probability if $t_0 = 3$.

(b) Can the error probability in part (a) be improved by choosing $t_0$ differently?

(c) Find the error probability using the following filter with $t_0 = 2$:

$$h(t) = \begin{cases} 1 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(d) Can you reduce the error probability in part (c) by sampling the filter output multiple times?

Problem 5. Let $\mathcal{W} = \{w_0(t), w_1(t)\}$ be the signal constellation used to communicate an equiprobable bit across an additive Gaussian noise channel. In this exercise, we verify that the projection of the channel output onto the inner product space $\mathcal{V}$ spanned by $\mathcal{W}$ is not necessarily a sufficient statistic, unless the noise is white.
Let $\psi_1(t), \psi_2(t)$ be an orthonormal basis for $\mathcal{V}$. We choose the additive noise to be $N(t) = Z_1\psi_1(t) + Z_2\psi_2(t) + Z_3\psi_3(t)$ for some normalized $\psi_3(t)$ that is orthogonal to $\psi_1(t)$ and $\psi_2(t)$, and choose $Z_1, Z_2, Z_3$ to be zero-mean jointly Gaussian random variables of identical variance $\sigma^2$. Let $c_i = (c_{i,1}, c_{i,2}, 0)^\mathsf{T}$ be the codeword associated to $w_i(t)$ with respect to the extended orthonormal basis $\psi_1(t), \psi_2(t), \psi_3(t)$. There is a one-to-one correspondence between the channel output $R(t)$ and $Y = (Y_1, Y_2, Y_3)^\mathsf{T}$, where $Y_i = \langle R, \psi_i \rangle$. In terms of $Y$, the hypothesis testing problem is

$$H = i : Y = c_i + Z, \quad i = \{0, 1\},$$

where we have defined $Z = (Z_1, Z_2, Z_3)^\mathsf{T}$.

(a) As a warm-up exercise, let us first assume that $Z_1, Z_2, Z_3$ are independent. Use the Fisher–Neyman factorization theorem to show that $(Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) Now assume that $Z_1$ and $Z_2$ are independent, but $Z_3 = Z_2$. Prove that in this case $(Y_1, Y_2)^\mathsf{T}$ is *not* a sufficient statistic.

(c) To check a specific case, consider $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$. Determine the error probability of an ML receiver that observes $(Y_1, Y_2)^\mathsf{T}$ and that of another ML receiver that observes $(Y_1, Y_2, Y_3)^\mathsf{T}$.

Problem 6. Let a channel output be

$$R(t) = cXw(t) + N(t), \tag{1}$$

where $c > 0$ is some deterministic constant, $X$ is a uniformly distributed random variable that takes values in $\{-3, -1, 1, 3\}$, $w(t) = \mathbb{1}_{[0,1)}(t)$, and $N(t)$ is white Gaussian noise of power spectral density $\frac{N_0}{2}$.

(a) Describe the receiver that, based on the channel output $R(t)$, decides on the value of $X$ with least probability of error.

(b) Find the error probability of the receiver in part (a).

(c) Suppose now that you still use the receiver in part (a), but that the received signal is actually

$$R(t) = \frac{3}{4}cXw(t) + N(t),$$

i.e., you were unaware that the channel was attenuating the signal. What is the probability of error now?

(d) Suppose now that you still use the receiver in part (a) and that $R(t)$ is according to (1), but that the noise is colored. In fact, $N(t)$ is a zero-mean stationary Gaussian noise process of auto-covariance function

$$K_N(\tau) = \frac{1}{4\alpha} e^{-|\tau|/\alpha},$$

where $0 < \alpha < \infty$ is some deterministic real parameter. What is the probability of error now?