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Handout 18 Information Theory and Coding
Homework 7 Nov. 3, 2015

Problem 1. A source produces independent, equally probable symbols from an alphabet (a_1, a_2) at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a_1 as 000 and the source symbol a_2 as 111. If in the corresponding 3 second interval of the channel output, any of the sequences 000,001,010,100 is received, a_1 is decoded; otherwise, a_2 is decoded. Let $\epsilon < 1/2$ be the channel crossover probability.

(a) For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a_1 came out of the source given that received sequence.

(b) Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.

(c) Find the probability of an incorrect decision (using part (a) is not the easy way here).

(d) If the source is slowed down to produce one letter every $2n + 1$ seconds, a_1 being encoded by $2n + 1$ 0's and a_2 being encoded by $2n + 1$ 1's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \to \infty$.

Problem 2. Consider two discrete memoryless channels. The first channel has input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$; the second channel has input alphabet $\mathcal{Y}$ and output alphabet $\mathcal{Z}$. The first channel is described by the conditional probabilities $P_1(y|x)$ and the second channel by $P_2(z|y)$. Let the capacities of these channels be C_1 and C_2. Consider a third memoryless channel described by probabilities

$$P_3(z|x) = \sum_{y \in \mathcal{Y}} P_2(z|y) P_1(y|x), \quad x \in \mathcal{X},\ z \in \mathcal{Z}.$$

(a) Show that the capacity C_3 of this third channel satisfies

$$C_3 \leq \min\{C_1, C_2\}.$$

(b) A helpful statistician preprocesses the output of the first channel by forming $\tilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.

(b1) Show that he is wrong.

(b2) Under what conditions does he not strictly decrease the capacity?

Problem 3. Let X be the channel input. Assume that the channel output Y is passed through a date processor in such a way that no information is lost. That is,

$$I(X;Y) = I(X;Z)$$

where Z is the processor output. Find an example where $H(Y) > H(Z)$ and find an example where $H(Y) < H(Z)$.
Hint: The data processor does not have to be deterministic

PROBLEM 4. Consider the discrete memoryless channel $Y = X + Z \pmod{11}$, where

$$\Pr(Z = 1) = \Pr(Z = 2) = \Pr(Z = 3) = 1/3$$

and $X \in \{0, 1, \dots, 10\}$. Assume that Z is independent of X.

(a) Find the capacity.

(b) What is the maximizing $p^*(x)$?

PROBLEM 5. We are given a memoryless stationary binary symmetric channel BSC(ϵ). Namely, if $X_1, \dots, X_n \in \{0, 1\}$ are the input of this channel and $Y_1, \dots, Y_n \in \{0, 1\}$ are the output, we have:

$$P(Y_i | X_i, X^{i-1}, Y^{i-1}) = P(Y_i | X_i) = \begin{cases} 1 - \epsilon & \text{if } Y_i = X_i, \\ \epsilon & \text{otherwise.} \end{cases}$$

Let W be a random variable that is uniform in $\{0, 1\}$ and consider a communication system with feedback which transmits the value of W to the receiver as follows:

- At time $t = 1$, the transmitter sends $X_1 = W$ through the channel.
- At time $t = i + 1 \leq n$, the transmitter gets the value of Y_i from the feedback and sends $X_{i+1} = Y_i$ through the channel.

(a) Give the capacity C of the channel in terms of ϵ, and show that $C = 0$ when $\epsilon = \frac{1}{2}$.

(b) Show that if $\epsilon = \frac{1}{2}$, $I(X^n; Y^n) = n - 1$. This means that $I(X^n; Y^n) \leq nC$ does not hold for this system.

(c) Show that although $I(X^n; Y^n) > nC$ when $\epsilon = \frac{1}{2}$, we still have $I(W; Y^n) \leq nC$.

Note that since W is the useful information that is being transmitted, it is the value of $I(W; Y^n)$ that we are interested in when we want to compute the amount of information that is shared with the receiver.

PROBLEM 6. Consider a random source $\mathcal{S}$ of information, and let W be a random variable which represents the first L symbols $U_1, \dots, U_L$ of this source, i.e., $W = U_1^L$. We want to transmit the value of W using a memoryless stationary channel as follows:

- At time $t = 1$, we send $X_1 = f_1(W)$ through the channel.
- At time $t = i + 1 \leq n$, we send $X_{i+1} = f_i(W, Y^i)$ through the channel. $Y_1, \dots, Y_i$ are the output of the channel at times $t = 1, \dots, i$ respectively,

$f_1, \dots, f_n$ are n mappings that constitute the encoder. Clearly, this is a communication system with feedback as we are using the value of Y^i in the computation of X_{i+1}.

In the previous problem, we gave an example which satisfies $I(X^n; Y^n) > nC$ and $I(W; Y^n) \leq nC$. Show that the inequality $I(W; Y^n) \leq nC$ always holds by justifying each of the following equalities and inequalities:

$$\begin{aligned} I(W; Y^n) &\overset{(a)}{=} \sum_{i=1}^{n} I(W; Y_i | Y^{i-1}) \overset{(b)}{\leq} \sum_{i=1}^{n} I(W, Y^{i-1}; Y_i) \overset{(c)}{\leq} \sum_{i=1}^{n} I(W, X_i, X^{i-1}, Y^{i-1}; Y_i) \\ &\overset{(d)}{=} \sum_{i=1}^{n} I(X_i, X^{i-1}, Y^{i-1}; Y_i) \overset{(e)}{=} \sum_{i=1}^{n} I(X_i; Y_i) \overset{(f)}{\leq} nC. \end{aligned}$$

Since $I(W; Y^n)$ represents the amount of information that is shared with the receiver, the inequality $I(W; Y^n) \leq nC$ shows that feedback does not increase the capacity.