Exercise Set 4 : 17-18 March 2016
Calcul Quantique

Exercise 1 *1-qubit non-classical gate*

(a) It follows from $G^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) For $|v\rangle = |0\rangle$ and $|1\rangle$, we have

$$\begin{aligned} G|0\rangle &= \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle \\ G^2|0\rangle &= |1\rangle \\ G|1\rangle &= \left(\frac{1}{2} - \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} + \frac{i}{2}\right)|1\rangle \\ G^2|1\rangle &= |0\rangle . \end{aligned}$$

So, for $|v\rangle = \alpha|0\rangle + \beta|1\rangle$, by linear combination we have

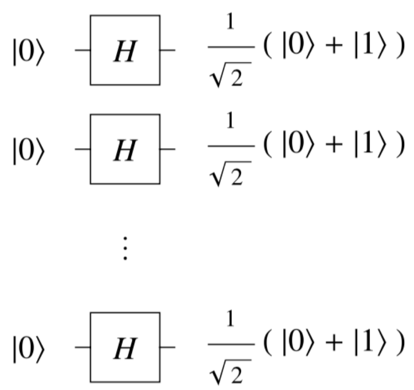
$$\begin{aligned} G|v\rangle &= \left((\alpha+\beta)\frac{1}{2} + (\alpha-\beta)\frac{i}{2}\right)|0\rangle + \left((\alpha+\beta)\frac{1}{2} - (\alpha-\beta)\frac{i}{2}\right)|1\rangle \\ G^2|v\rangle &= \alpha|1\rangle + \beta|0\rangle . \end{aligned}$$

(c) NOT gate

Exercise 2 *Superposition of exponentially many component states*

The Hadamard gate H transform $|0\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. By concatenating all the outputs from the Hadarmard gates (see the below figure), the resulting tensor product state is

$$\begin{aligned} H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2^n}} \sum_{b_1, b_2, \ldots, b_m \in \{0,1\}^2} |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle . \end{aligned}$$



Exercise 3 *SWAP · controlled-U · SWAP*

(a) Let $|t\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Then, we have

$$|0\rangle \otimes |t\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\0\\0\end{pmatrix}, \qquad |1\rangle \otimes |t\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\0\\ \alpha\\ \beta\end{pmatrix}.$$

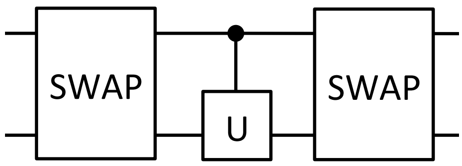
Now we verify the output of controlled-U with the given matrix representation indeed gives $|c\rangle \otimes U^c|t\rangle$:

$$(\text{controlled-}U)\,|0\rangle \otimes |t\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&U_{11}&U_{12}\\0&0&U_{21}&U_{22}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\0\\0\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = |0\rangle \otimes |t\rangle$$

and

$$\begin{aligned}(\text{controlled-}U)\,|1\rangle \otimes |t\rangle &= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&U_{11}&U_{12}\\0&0&U_{21}&U_{22}\end{pmatrix}\begin{pmatrix}0\\0\\ \alpha\\ \beta\end{pmatrix}\\ &= \begin{pmatrix}0\\0\\U_{11}\alpha + U_{12}\beta\\U_{21}\alpha + U_{22}\beta\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}U_{11}&U_{12}\\U_{21}&U_{22}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = |1\rangle \otimes U|t\rangle.\end{aligned}$$

(b) The circuit for SWAP · controlled-U · SWAP :



(c) We have seen the matrix representation of controlled-SWAP in Homework 2. Similarly, SWAP has the matrix representation

$$\text{SWAP} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Then we have

$$\text{SWAP} \cdot \text{controlled-}U \cdot \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & 1 & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix}$$

Exercise 4 *Contolled-controlled-U*

We check all the input cases :

$$\begin{aligned}
|000\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |000\rangle \\
|001\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |001\rangle \\
|010\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |01\rangle \otimes (V^\dagger \cdot V\,|0\rangle) = |010\rangle \\
|011\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |01\rangle \otimes (V^\dagger \cdot V\,|1\rangle) = |011\rangle \\
|100\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |10\rangle \otimes (V^\dagger \cdot V\,|0\rangle) = |100\rangle \\
|101\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |10\rangle \otimes (V^\dagger \cdot V\,|1\rangle) = |101\rangle \\
|110\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |11\rangle \otimes (V^2\,|0\rangle) = |11\rangle \otimes U\,|0\rangle \\
|111\rangle &\xrightarrow{\text{ctrl-ctrl-}U} |11\rangle \otimes (V^2\,|1\rangle) = |11\rangle \otimes U\,|1\rangle
\end{aligned}$$

■