

Exercise 1 *Production of Bell states*

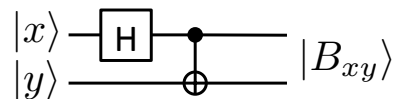
a) Direct computation gives

$$\begin{aligned}
 (CNOT)(H \otimes I) |x\rangle \otimes |y\rangle &= (CNOT)H |x\rangle \otimes |y\rangle \\
 &= (CNOT) \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle) \otimes |y\rangle \\
 &= \frac{1}{\sqrt{2}} CNOT |0, y\rangle + \frac{(-1)^x}{\sqrt{2}} CNOT |1, y\rangle \\
 &= \frac{1}{\sqrt{2}} |0, y\rangle + \frac{(-1)^x}{\sqrt{2}} |1, y \oplus 1\rangle
 \end{aligned}$$

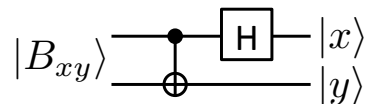
More explicitly, we enumerate all the cases :

$$\begin{aligned}
 (CNOT)(H \otimes I) |00\rangle &= (CNOT) \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |B_{00}\rangle \\
 (CNOT)(H \otimes I) |01\rangle &= (CNOT) \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) = |B_{01}\rangle \\
 (CNOT)(H \otimes I) |10\rangle &= (CNOT) \frac{1}{\sqrt{2}} (|00\rangle - |10\rangle) = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = |B_{10}\rangle \\
 (CNOT)(H \otimes I) |11\rangle &= (CNOT) \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) = |B_{11}\rangle
 \end{aligned}$$

b) The circuit corresponding to $|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle$:



c) The circuit corresponding to $|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$:



Exercise 2 *Construction of a multi-control- U .*

We show the quantum state at each stage of the circuit.

$$\begin{aligned}
 & \text{Input : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes |c_t\rangle \\
 & \text{After the 1st Toffoli gate : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes |c_t\rangle \\
 & \text{After the 2nd Toffoli gate : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes |c_t\rangle \\
 & \text{After the controlled-}U \text{ gate : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |c_1 \cdot c_2 \cdot c_3\rangle \otimes U^{c_1 c_2 c_3} |c_t\rangle \\
 & \text{After the 3rd Toffoli gate : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |c_1 \cdot c_2\rangle \otimes |0\rangle \otimes U^{c_1 c_2 c_3} |c_t\rangle \\
 & \text{After the 4th Toffoli gate : } |c_1\rangle \otimes |c_2\rangle \otimes |c_3\rangle \otimes |0\rangle \otimes |0\rangle \otimes U^{c_1 c_2 c_3} |c_t\rangle
 \end{aligned}$$

Exercise 3 *Construction of the Toffoli gate from a control-NOT (Indication : long calculation).*

Note that $(CNOT) |x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle$ can also be represented as $|x\rangle \otimes X^x |y\rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is one of the Pauli gates (see Chapter 3). Therefore, the circuit outputs the tensor product state $|\psi\rangle$ given by

$$|\psi\rangle = T |c_1\rangle \otimes S X^{c_1} T^\dagger X^{c_1} T^\dagger |c_2\rangle \otimes H T X^{c_1} T^\dagger X^{c_2} T X^{c_1} T^\dagger X^{c_2} H |t\rangle.$$

We then verify explicitly all the cases of c_1 and c_2 . The calculation largely uses the fact that all the quantum gates here are unitary (*e.g.*, $TT^\dagger = T^\dagger T = I$); in particular, the gates X and H are involutory, *i.e.*, $X^2 = H^2 = I$.

For $c_1 = 0$, we have

$$\begin{aligned}
 |\psi\rangle &= T |0\rangle \otimes S T^\dagger T^\dagger |c_2\rangle \otimes H T T^\dagger X^{c_2} T T^\dagger X^{c_2} H |t\rangle \\
 &= |0\rangle \otimes |c_2\rangle \otimes H (T T^\dagger) (X^{c_2} (T T^\dagger) X^{c_2}) H |t\rangle \\
 &= |0\rangle \otimes |c_2\rangle \otimes |t\rangle \\
 &= |0\rangle \otimes |c_2\rangle \otimes |t \oplus 0 \cdot c_2\rangle
 \end{aligned}$$

For $c_1 = 1$ and $c_2 = 0$, we calculate

$$X T^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{-i\pi/4} T$$

and therefore we have

$$\begin{aligned}
 |\psi\rangle &= T |1\rangle \otimes S X T^\dagger X T^\dagger |0\rangle \otimes H T X T^\dagger T X T^\dagger H |t\rangle \\
 &= e^{i\pi/4} |1\rangle \otimes S (X T^\dagger X) T^\dagger |0\rangle \otimes H (T (X (T^\dagger T) X) T^\dagger) H |t\rangle \\
 &= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} S T T^\dagger |0\rangle \otimes |t\rangle \\
 &= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} |0\rangle \otimes |t\rangle \\
 &= |1\rangle \otimes |0\rangle \otimes |t\rangle \\
 &= |1\rangle \otimes |0\rangle \otimes |t \oplus 1 \cdot 0\rangle.
 \end{aligned}$$

Finally, for $c_1 = c_2 = 1$, we calculate

$$(TXT^\dagger X)^2 = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{-i\pi/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e^{-i\pi/2} Z,$$

$$HZH = X$$

and therefore we have

$$\begin{aligned} |\psi\rangle &= T |1\rangle \otimes SXT^\dagger XT^\dagger |1\rangle \otimes HTXT^\dagger XTXT^\dagger XH |t\rangle \\ &= e^{i\pi/4} |1\rangle \otimes S (XT^\dagger X) T^\dagger |1\rangle \otimes H (TXT^\dagger X)^2 H |t\rangle \\ &= e^{i\pi/4} |1\rangle \otimes e^{-i\pi/4} S T T^\dagger |1\rangle \otimes e^{-i\pi/2} H Z H |t\rangle \\ &= e^{i\pi/4} |1\rangle \otimes e^{i\pi/4} |1\rangle \otimes e^{-i\pi/2} X |t\rangle \\ &= |1\rangle \otimes |1\rangle \otimes X |t\rangle \\ &= |1\rangle \otimes |1\rangle \otimes |t \oplus 1\rangle. \end{aligned}$$

Exercise 4 *Unitary representation of a reversible computation.*

- a) The relevant Hilbert space is $(\mathbb{C}^2)^{\otimes(n+1)}$. A unitary matrix is precisely a matrix that conserves the scalar product. Therefore, to show U_f is a unitary matrix, we show it conserves the scalar product. For any two states $|x_1, \dots, x_n; y\rangle$ and $|\hat{x}_1, \dots, \hat{x}_n; \hat{y}\rangle$, we have

$$\begin{aligned} \langle x_1, \dots, x_n; y | U_f^\dagger U_f | \hat{x}_1, \dots, \hat{x}_n; \hat{y} \rangle &= \langle x_1, \dots, x_n; y \oplus f(x_1, \dots, x_n) | \hat{x}_1, \dots, \hat{x}_n; \hat{y} \oplus f(\hat{x}_1, \dots, \hat{x}_n) \rangle \\ &= \begin{cases} 1 & \text{if } x_i = \hat{x}_i \text{ for all } i \text{ and } y = \hat{y} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which is equivalent to $\langle x_1, \dots, x_n; y | \hat{x}_1, \dots, \hat{x}_n; \hat{y} \rangle$.

- b) When the output of f is in $\{0, 1\}^m$, so we need m storage bits y_1, \dots, y_m . The relevant Hilbert space would be $(\mathbb{C}^2)^{\otimes(n+m)}$. The same calculation in (a) gives

$$\langle x_1, \dots, x_n; y_1, \dots, y_m | U_f^\dagger U_f | \hat{x}_1, \dots, \hat{x}_n; \hat{y}_1, \dots, \hat{y}_m \rangle = \langle x_1, \dots, x_n; y_1, \dots, y_m | \hat{x}_1, \dots, \hat{x}_n; \hat{y}_1, \dots, \hat{y}_m \rangle.$$

We conclude that U_f is unitary.