
Exercise Set 1 : 25-26 February 2016
Calcul Quantique

Exercise 1 *Dirac's notation for vectors and matrices*

(a) If $|w\rangle$ is a vector and α is a scalar, then

$$(\alpha |w\rangle)^\dagger = \langle w| \alpha^* = \alpha^* \langle w|$$

(you can check this in components). Moreover, we have linearity :

$$(\alpha |v\rangle + \beta |w\rangle)^\dagger = (\alpha |v\rangle)^\dagger + (\beta |w\rangle)^\dagger.$$

Then we get

$$\begin{aligned} \langle v| = (|v\rangle)^\dagger &= (v_1 |e_1\rangle + v_2 |e_2\rangle + \dots + v_N |e_N\rangle)^\dagger \\ &= v_1^* \langle e_1| + v_2^* \langle e_2| + \dots + v_N^* \langle e_N|. \end{aligned}$$

(b) If $\langle v| = \sum_{i=1}^N v_i^* \langle e_i|$ and $|w\rangle = \sum_{j=1}^N w_j |e_j\rangle$, then

$$\begin{aligned} \langle v|w\rangle &= \sum_{i=1}^N \sum_{j=1}^N v_i^* w_j \langle e_i|e_j\rangle \\ &= \sum_{i=1}^N \sum_{j=1}^N v_i^* w_j \delta_{ij} \\ &= \sum_{i=1}^N v_i^* w_i. \end{aligned}$$

(c) Same method provided in (a).

(d) For $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we have $\|\vec{v}\|^2 = \vec{v}^{T,*} \cdot \vec{v}$, so $\|\vec{v}\|^2 = \alpha^* \alpha + \beta^* \beta$. On the other hand, $\langle v|v\rangle = \alpha^* \alpha + \beta^* \beta$ also by (b).

(e) Write the ket-bra $\langle e_k| A |e_l\rangle$ as a matrix to realize that

$$|e_k\rangle \langle e_l| = \text{\textit{k-th pos}} \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \underbrace{(0 \dots 0 1 0 \dots 0)}_{\text{\textit{l-th pos}}} = k \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \right.$$

Thus,

$$A = \sum_{k,l} a_{kl} |e_k\rangle \langle e_l|.$$

So,

$$\begin{aligned} \langle e_i | A | e_j \rangle &= \sum_{l,k} a_{kl} \langle e_i | e_k \rangle \langle e_l | e_j \rangle \\ &= \sum_{l,k} a_{kl} \delta_{ik} \delta_{lj} \\ &= a_{ij}. \end{aligned}$$

(f) From the beginning of point (e), we have

$$I = \sum_{i=1}^N |e_i\rangle \langle e_i|.$$

Indeed, $|e_i\rangle \langle e_i|$ is the matrix with 1 at the i -th row and i -th column and zeros elsewhere. This is called the closure relation.

(g) First note that the closure relation is valid for any orthonormal basis. Indeed, if $\{|\varphi_i\rangle\}_{i=1\dots N}$ are orthonormal, there exists a unitary basis change (a “rotation”) such that

$$\begin{aligned} |\varphi_i\rangle &= U |e_i\rangle, \\ \langle \varphi_i| &= \langle e_i| U^\dagger. \end{aligned}$$

Then from $I = \sum_{i=1}^N |e_i\rangle \langle e_i|$ we get :

$$\begin{aligned} UIU^\dagger &= \sum_{i=1}^N U |e_i\rangle \langle e_i| U^\dagger \\ I &= \sum_{i=1}^N |\varphi_i\rangle \langle \varphi_i|. \end{aligned}$$

Now, from $\alpha_i |\varphi_i\rangle = A |\varphi_i\rangle$ we get

$$\sum_{i=1}^N \alpha_i |\varphi_i\rangle \langle \varphi_i| = \sum_{i=1}^N A |\varphi_i\rangle \langle \varphi_i| = A \sum_{i=1}^N |\varphi_i\rangle \langle \varphi_i| = AI = A.$$

Exercise 2 Tensor Product in Dirac’s notation

(a) By distributivity of the tensor product (first two properties), it follows that :

$$\begin{aligned} |v\rangle_1 \otimes |w\rangle_2 &= \left(\sum_{i=1}^N v_i |e_i\rangle_1 \right) \otimes \left(\sum_{j=1}^M w_j |f_j\rangle_2 \right) \\ &= \sum_{i=1}^N \sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2. \end{aligned}$$

(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then by definition of the inner product :

$$\langle e_k, f_l | e_i, f_j \rangle = \langle e_k | e_i \rangle \langle f_l | f_j \rangle = \delta_{kl} \cdot \delta_{lj} = \delta_{(k,l);(l,j)}.$$

So this equals one if and only if $(k, l) = (i, j)$ and zero otherwise. This means that $\{|e_i, f_j\rangle; i = 1 \dots N; j = 1 \dots N\}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The dimension equals the number of basis vectors, so is NM , the product of $\dim \mathcal{H}_1$ and $\dim \mathcal{H}_2$.

(c) We apply the definition

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2$$

to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i, j) = (k, l)$ and 0 otherwise. This means :

$$A \otimes B |e_k, f_l\rangle = A |e_k\rangle \otimes B |f_l\rangle$$

and multiplying by $\langle e_i, f_j |$, we find :

$$\begin{aligned} \langle e_i, f_j | A \otimes B |e_k, f_l\rangle &= (\langle e_i | \otimes \langle f_j |) (A |e_k\rangle \otimes B |f_l\rangle) \\ &= \langle e_i | A |e_k\rangle \langle f_j | B |f_l\rangle \\ &= a_{ik} b_{jl}. \end{aligned}$$

■