

---

Exercise Set 3 : 10-11 March 2016  
Calcul Quantique

---

**Exercise 1** *Production of Bell states*

a) Check the following identity using Dirac's notation :

$$|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle.$$

where  $x, y \in \{0, 1\}$  and  $|B_{xy}\rangle$  are the Bell states.

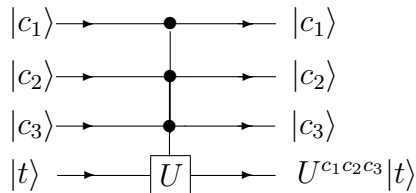
b) Represent the corresponding circuit.

c) Represent the circuit corresponding to the inverse identity :

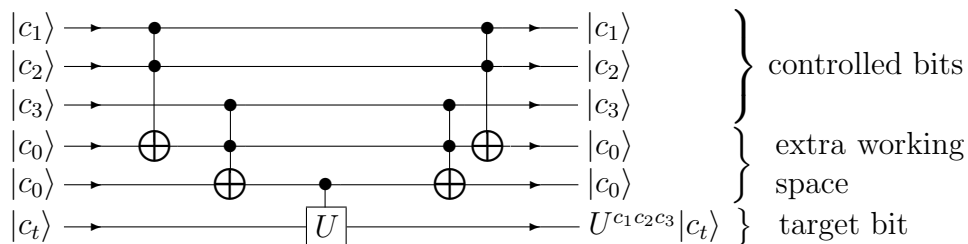
$$|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$$

**Exercise 2** *Construction of a multi-control-U.*

Verify that the multi-control- $U$  :

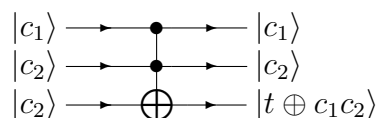


can be realized with the Toffoli gate (control-control-NOT) a simple control- $U$ .

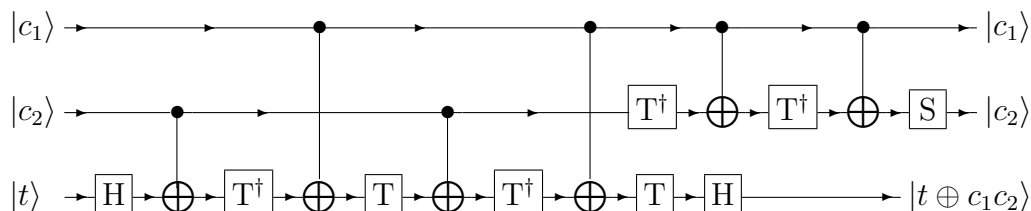


**Exercise 3** Construction of the Toffoli gate from a control-NOT (Indication : long calculation).

Verify that the control-control-NOT also called Toffoli gate :



is equivalent to the following circuit made of *CNOT*, *H*, *T* and *S*



**Exercise 4** Unitary representation of a reversible computation.

Classically a Boolean function  $f$  with inputs  $(x_1, \dots, x_n) \in \{0, 1\}^n$  and output in  $\{0, 1\}$  can be computed reversibly as

$$\tilde{f}(x_1, \dots, x_n; y) = (x_1, \dots, x_n; y \oplus f(x_1, \dots, x_n))$$

where  $y \in \{0, 1\}$  is a single storage bit.

This can be implemented in a quantum circuit thanks to the following unitary operation

$$U_f |x_1, \dots, x_n; y\rangle = |x_1, \dots, x_n; y \oplus f(x_1, \dots, x_n)\rangle$$

a) What is the Hilbert space relevant for this implementation. Prove that  $U_f$  is indeed a unitary matrix.

b) Generalize this discussion to the case where the output of  $f$  in  $\{0, 1\}^m$  (there are  $m$  output bits).