Exercise Set 1 : 25-26 February 2016
Calcul Quantique

Exercise 1 *Dirac's notation for vectors and matrices*

Let $\mathcal{H} = \mathbb{C}^N$ be a vector space of N dimensional vectors with complex components. If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$ is a column vector, we define its conjugate as $\vec{v}^\dagger = \vec{v}^{T,*} = (v_1^*, \ldots, v_N^*)$ where * is complex conjugate. The inner or scalar product is $\vec{v}^\dagger \cdot \vec{w} = v_1^* w_1 + \cdots + v_N^* w_N$. In Dirac's notation we write $\vec{v} = |v\rangle$ and $\vec{v}^\dagger = \langle v|$. The canonical orthonormal basis vectors are written as $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. Thus $|v\rangle = v_1 |e_1\rangle + v_2 |e_2\rangle + \cdots + v_N |e_N\rangle$. The inner product of basis vectors is $\langle e_i | e_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$

(a) Check that if $|v\rangle = v_1 |e_1\rangle + v_2 |e_2\rangle + \cdots + v_N |e_N\rangle$ then

$$\langle v| = v_1^* \langle e_1| + v_2^* \langle e_2| + \cdots + v_N^* \langle e_N| \,.$$

(b) Deduce in Dirac notation

$$\langle v|w\rangle = v_1^* w_1 + \cdots + v_N^* w_N.$$

(c) Check that if $|v\rangle = \alpha |v'\rangle + \beta |v''\rangle$ then

$$\langle v| = \alpha^* \langle v'| + \beta^* \langle v''| \,.$$

(d) Show that $\sqrt{\langle v|v\rangle} = \|v\|$, the norm of $\vec{v}$ or $|v\rangle$.

(e) Consider an $N \times N$ matrix A with complex matrix element a_{ij} ; $i = 1 \ldots N$; $j = 1 \ldots N$. Show that

$$a_{ij} = \langle e_i| A |e_j\rangle \,.$$

(f) Show that the identity matrix satisfies :

$$I = \sum_{i=1}^{N} |e_i\rangle \langle e_i| \,.$$

This is called the closure relation.

(g) (Spectral theorem) Let $A = A^\dagger$ where $A^\dagger = A^{T,*}$ be a hermitian matrix. It has N orthonormal eigenvectors with real eigenvalues. Let $|\varphi_i\rangle, \alpha_i$ be the eigenvectors and eigenvalues of A, *i.e.*,

$$A|\varphi_i\rangle = \alpha_i |\varphi_i\rangle .$$

Show that

$$A = \sum_{i=1}^{N} \alpha_i |\varphi_i\rangle \langle\varphi_i| .$$

This is called the spectral theorem.
Hint : consider $\langle e_i| A |e_j\rangle$, use the eigenvalue equation and the closure relation.

Exercise 2 *Tensor Product in Dirac's notation*

Let $\mathcal{H}_1 = \mathbb{C}^N$ and $\mathcal{H}_2 = \mathbb{C}^M$ be N and M dimensional Hilbert spaces. The tensor product space $\mathcal{H}_1 \bigotimes \mathcal{H}_2$ is a new Hilbert space formed by "pairs of vectors" denoted as $|v\rangle_1 \otimes |w\rangle_2 \equiv |v, w\rangle$ with the properties :

- $\big(\alpha |v\rangle_1 + \beta |v'\rangle_1\big) \otimes |w\rangle_2 = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v'\rangle_1 \otimes |w\rangle_2$,
- $|v\rangle_1 \otimes \big(\alpha |w\rangle_2 + \beta |w'\rangle_2\big) = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v\rangle_1 \otimes |w'\rangle_2$,
- $\big(|v\rangle_1 \otimes |w\rangle_2\big)^\dagger = \langle v|_1 \otimes \langle w|_2$,
- $\langle v, w|v', w'\rangle = \langle v|v'\rangle_1 \langle w|w'\rangle_2$.

(a) Show that for any basis of $|v\rangle_1 = \sum_{i=1}^{N} v_i |e_i\rangle_1$ and $|w\rangle_2 = \sum_{j=1}^{M} w_j |f_j\rangle_2$ then

$$|v\rangle_1 \otimes |w\rangle_2 = \sum_{i=1}^{N} \sum_{j=1}^{M} v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2 .$$

(b) Show that if $\{|e_i\rangle_1 ; i = 1 \dots N\}$ and $\{|f_j\rangle_2 ; j = 1 \dots M\}$ are orthonormal, then $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle$ is an orthonormal basis of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. What is the dimension of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$?

(c) Any vector $|\Psi\rangle$ of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$ can be expanded by the basis $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle, i = 1 \dots N, j = 1 \dots M$,

$$|\Psi\rangle = \sum_{i=1,j=1}^{N,M} \psi_{ij} |e_i, f_j\rangle .$$

If A is a matrix acting on $\mathcal{H}_1$ and B is a matrix acting on $\mathcal{H}_2$, the tensor product $A \otimes B$ is defined as

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2 .$$

Check that the matrix elements of $A \otimes B$ in the basis $|e_i, f_j\rangle$ are :

$$\langle e_i, f_j| A \otimes B |e_k, f_l\rangle = a_{ik} b_{kl}.$$

(d) Let $\mathcal{H}_1 = \mathbb{C}^2$, $\mathcal{H}_2 = \mathbb{C}^2$. Take $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $|v\rangle_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|w\rangle_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$.
Check the following :

$$|v\rangle_1 \otimes |w\rangle_2 = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}, \qquad A_1 \otimes B_2 = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.$$

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