Problem 1 (Distribution of cavity fields in the TAP theory.)**.** The goal of this exercise is to numerically justify some of the heuristic arguments of this chapter. When we discuss state evolution for compressive sensing we will encounter similar arguments and hopefully these will seem familiar. Consider the SK model with i.i.d Bernoulli(1/2) coupling constants $\tilde{J}_{ij} = \pm 1$, or with $\tilde{J}_{ij}$ Gaussian with zero mean and unit variance. The TAP approximation to the BP equations reads

$$m_j^{(\ell)} = \tanh\{\beta(h + \sum_{i \neq j} \hat{h}_{i \to j}^{(\ell)})\}$$

where the update of the cavity fields is

$$\hat{h}_{i \to j}^{(\ell)} = \frac{1}{\sqrt{n}} \tilde{J}_{ij} m_i^{(\ell-1)} - \frac{\beta}{n} m_j^{(\ell-1)} (1 - (m_i^{(\ell-1)})^2)$$

and the initialization $\hat{h}_{i \to j}^{(0)} = 0$.

Take a number $N = 50$ of realizations (coupling constants) of the system of size $n = 500$ or 1000 and an iteration number say $\ell = 10$. Try values of $(h, T = \beta^{-1})$ in the high temperature regime. The following should be suitable $(h = 0.5, T = 1.2)$ and $(h = 1, T = 0.8)$.

(i) Plot the histogram of the total cavity field

$$\hat{h}_{\text{cav}}^{(\ell)} = \sum_{i \neq j} \hat{h}_{i \to j}^{(\ell)}.$$

This field is equal to a "Curie-Weiss" field to which the "Onsager reaction term" is subtracted. Plot the histogram of the total Curie-Weiss contribution

$$h_{\text{CW}}^{(\ell)} = \sum_{i \neq j} \frac{1}{\sqrt{n}} \tilde{J}_{ij} m_i^{(\ell-1)}.$$

(ii) Check that the Edwards-Anderson parameter

$$q^{(\ell)} = \frac{1}{n} \sum_{i=1}^{n} (m_i^{(\ell)})^2.$$

is concentrated on its empirical mean over the N realizations.

(iii) Compare both histograms with the Gaussian distribution of zero mean and variance equal to the Edwards-Anderson parameter. You should observe that the histogram of the cavity field agrees with this Gaussian.