ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 Statistical Physics for Communication and Computer Science

Homework 6 April 1, 2015

Problem 1 (Belief Propagation for $(3,6)$ Ensemble and AWGN Channel)**.** In the first homework you have implemented a program which can generate random elements from a regular Gallager ensemble. We will now use this, together with the message-passing algorithm discussed in class, to simulate transmission over a BAWGN channel.

We will use elements from the $(3,6)$-ensemble of length $n = 1024$. For every codeword we send we generate a new code. This way we get the so called *ensemble average*. As discussed in class last week, when transmitting with a binary linear code over a symmetric channel, we can in fact assume that the all-zero (in 0/1 notation) codeword was sent since the error probability is independent of the transmitted codeword. This simplifies our life since we do not need to implement an encoder. We assume that we send the codeword over a binary-input additive white Gaussian noise channel. More precisely, the input is ± 1 (with the usual mapping). The channel adds to each component of the codeword an independent Gaussian random variable with zero mean and variance σ^2. At the receiver implement the message-passing decoder discussed in class. It is typically easiest to do the computations with likelihoods. Since a random element from the $(3,6)$ ensemble typically does not have a tree-like factor graph the scheduling of the messages is important. To be explicit, assume that we use a *parallel* schedule. This means, we start by sending all *initial* messages from variable nodes to check nodes. We then process these messages and send messages back from check nodes to all variable nodes. This is one *iteration.* For each codeword perform 100 iterations and then make the final decision for each bit.

Plot the negative logarithm (base 10) of the resulting bit error probability as a function of the capacity of the BAWGN channel with variance σ^2. This capacity does not have a closed form but can be computed by means of the numerical integral

$$C(\sigma^2) = \int_{-1}^{1} \frac{\sigma}{\sqrt{2\pi}(1-y^2)} e^{-\frac{(1-\sigma^2\tanh^{-1}(y))^2}{2\sigma^2}} \log_2(1+y)dy.$$

If the code and the decoder where optimal and the length of the code were infinite, where should you see the phase transition (rapid decay of error probability)?

Problem 2 (Gallager Algorithm A)**.** In class we discussed the BP algorithm which is the "locally optimal" message-passing algorithm. One of its downsides in a practical application is that it requires the exchange of real numbers. Hence, in any implementation messages are quantized to a fixed number of bits. One way to think of such a quantized algorithm is that the message represents an "approximation" of the underlying message that BP would have sent.

Assume that we are limited to exchange messages consisting of a single bit. Recall that for BP a positive message means that our current estimate of the associated bit is $+1$, whereas a negative message means that our current estimate is -1 (the magnitude of the BP message conveys our certainty). So we can think of a message-passing algorithm which is limited to exchange messages consisting of a single bit, as exchanging only the sign of their estimate.

The best known such algorithm (and historically also the oldest) is Gallager's algorithm A. It has the following message passing rules.

We assume that the codewords and the received word have components in $\{0,1\}$.

(i) *Initialization:* In the first iteration send out the received bits along all edges incident to a variable node.

(ii) *Check Node Rule:* At a check node send out along edge e the XOR of the incoming messages (not counting the incoming message along edge e).

(iii) *Variable Node Rule:* At a variable node. Send out the received value along edge e unless all incoming messages (not counting the incoming message on edge e) all agree in their value. Then send this value.

Assume that transmission takes place over the BSC(p) and that we are using a $(3,6)$-regular Gallager ensemble. Write down the density evolution equations for the Gallager algorithm A. What is the threshold for this combination?