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Handout 4 Statistical Physics for Communication and Computer Science
Homework 4 March 18, 2015

In homework 2 you proved that the Ising model in one dimension ($d = 1$) does not have a phase transition for any $T > 0$. On the grid $\mathbb{Z}^d$ there is a non trivial phase diagram with first and second order phase transitions for any $d \geq 2$. This is also the case on the complete graph (as shown in the lectures) which morally corresponds to $d = +\infty$. Another graph that in a sense, corresponds to $d = +\infty$, is the q-ary tree for $q \geq 3$. Indeed on $\mathbb{Z}^d$ the number of lattice sites at distance less than n from the origin scales as n^d. On the q-ary tree it scales as $(q-1)^n$ which grows faster than n^d for any finite d (for $q \geq 3$). Of course $q = 2$ corresponds to $\mathbb{Z}_+$.

The goal of the two exercises below is to solve for the Ising model on a q-ary tree and show that it displays first and second order phase transitions (with similar qualitative properties than on a complete graph).

Consider a finite rooted tree and call the root vertex o. All vertices have degree q, except for the leaf nodes that have degree 1. We suppose that the tree has n levels (the root being "level 0"). The thermodynamic limit corresponds to $n \to +\infty$. The Hamiltonian (multiplied by β) is

$$\beta\mathcal{H}_n = -K \sum_{(i,j)\in E_n} s_i s_j - h \sum_{i\in V_n} s_i \tag{1}$$

were $K > 0$, $h \in \mathbb{R}$, V_n is the set of vertices and E_n the set of edges. We are interested in the magnetization *of the root node* in the thermodynamic limit:

$$m(K,h) = \lim_{n\to+\infty} < s_o >_n = \frac{\sum_{\{s_k, k\in V_n\}} s_o \, e^{-\beta\mathcal{H}_n}}{Z_n} \tag{2}$$

The formula $\tanh^{-1} y = \frac{1}{2}\ln\frac{1+y}{1-y}$ might be useful.

Problem 1 (Recursive equations)**.** Perform the sums over the spins attached at the leaf nodes and show that

$$< s_o >_n = \frac{\sum_{\{s_k, k\in V_{n-1}\}} s_o \, e^{-\beta\mathcal{H}'_{n-1}}}{Z'_{n-1}} \tag{3}$$

where E_{n-1} and V_{n-1} are the edge and vertex sets of a tree with with $n-1$ levels and the new Hamiltonian is

$$\beta\mathcal{H}'_n = -K \sum_{(i,j)\in E_{n-1}} s_i s_j - h \sum_{i\in V_{n-1}} s_i - (q-1)\tanh^{-1}(\tanh K \tanh h) \sum_{i\in \text{level } n-1} s_i \tag{4}$$

Iterate this calculation and deduce

$$< s_o >_n = \tanh(h + q\tanh^{-1}(\tanh K \tanh u_n)) \tag{5}$$

where

$$u_{k+1} = h + (q-1)\tanh^{-1}(\tanh K \tanh u_k), \qquad u_1 = h \tag{6}$$

Check that for $q = 2$ you get back the recursion of homework 2.

Problem 2 (Analysis of the recursion)**.** We want to analyze the fixed point equation for $q \geq 3$,

$$u = h + (q-1)\tanh^{-1}(\tanh K \tanh u) \tag{7}$$

Plot the curves $u \to u - h$ and $u \to (q-1)\tanh^{-1}(\tanh K \tanh u)$ and show that:

- for $K \leq K_c \equiv \frac{1}{2}\ln\frac{q}{q-2} = \tanh^{-1}(q-1)^{-1}$, (7) has a unique solution, and that the iterations (6) converge to this unique solution.
- for $K > K_c$:
  - for $|h| \geq h_s$, (7) has a unique solution (you do not needw3 to compute h_s explicitly although it is possible to find its analytical expression) and that the iterations (6) converge to this unique solution.
  - for $|h| < h_s$, (7) has three solutions $u_-(h) < u_0(h) < u_+(h)$. Check graphically that for $h > 0$ the iterations (6) with initial condition $u_1 = h$ converge to $u_+(h)$. Similarly for $h < 0$ they converge to $u_-(h)$. Check also graphically that the fixed point $u_0(h)$ is unstable whereas $u_\pm(h)$ are stable.

Problem 3 (Phase transitions)**.** Now we want to discuss the consequences of the results in problem 2 for the phase diagram. On a tree the magnetization is defined as the average spin of the root

$$m(K,h) = \lim_{n\to+\infty}\langle s_o\rangle_n, \tag{8}$$

and we define the "spontaneous magnetization" as $m_\pm(K) = \lim_{h\to 0_\pm} m(K,h)$.
You will show that in the (K^{-1}, h) plane there is a first order phase transition line $(K^{-1} \in [0, K_c^{-1}[\,, h = 0)$ terminated by a critical point K_c. Outside of this line $m(K,h)$ is an analytic function of each variable.

- Deduce from the analysis in problem 2 that for $K \leq K_c$, $m_+(K) = m_-(K) = 0$.
- Deduce that for $K > K_c$, $m_+(K) \neq m_-(K)$ (jump discontinuity or first order phase transition) and that for $K \to +\infty$ $m_\pm \to \pm 1$.
- Show that for $K \to K_c$ from above, $m_\pm(K) \sim (K - K_c)^{1/2}$. So on the line $h = 0$, as a function of K, the spontaneous magnetization is continuous but not differentiable at K_c (second order phase transition).
- Now fix $K = K_c$ and show that $m(K_c, h) \sim |h|^{1/3}$. As a function of h the spontaneous magnetization is continuous but not differentiable at K_c (second order phase transition).

Hint: for the last two questions you can expand the fixed point equation to order u^3.

Remark 1: Note that the exponents 1/2 and 1/3 are the same than for the model on a complete graph. This is also the case for all $d \geq 4$ and is not the case for $d = 2$, 3.

Remark 2: On a tree the definition of the magnetization above is *not equivalent* to minus the derivative of the free energy with respect to h. In fact there is a fine point: $-\frac{1}{n}\ln Z_n$ is dominated by the contributions of leaf nodes and is not the "physically meaningful" definition of free energy. Rather the "physically meaningful" definition is given by an integral with respect to h of the magnetization at the root