ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 16 Principles of Digital Communications
Midterm Apr. 17, 2015

3 problems, 35 points, 165 minutes, closed book
Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET

Problem 1. (12 points) A symbol $X \in \{+1, -1\}$ with $\Pr(X = +1) = p$ is transmitted through two channels simultaneously. The outputs of the channels are

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2.$$

Here Z_1 is a random variable that *depends* on X: if $X = +1$, Z_1 is uniformly distributed on the interval $[-1, +1]$, and if $X = -1$, Z_1 is uniformly distributed on the interval $[-2, +2]$. The random variable Z_2 is independent of both X and Z_1 and is $\mathcal{N}(0, 1)$.

(a) (3 pts) Consider a receiver that only observes Y_2. Describe the MAP rule this receiver should implement to estimate X.

(b) (3 pts) Consider now a receiver that observes (Y_1, Y_2). Show that $T = (U, Y_2)$ with

$$U = \begin{cases} -1 & Y_1 < 0 \\ 0 & 0 \le Y_1 \le 1 \\ +1 & Y_1 > 1 \end{cases}$$

is a sufficient statistic to estimate X.

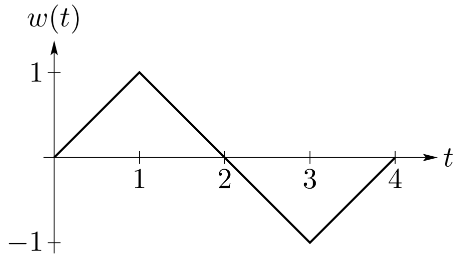
(c) (3 pts) Sketch the decision regions that minimize the probability of error to estimate X on the (y_1, y_2) plane.

(d) (3 pts) Express the probability of error in terms of p and the Q function.

PROBLEM 2. (11 points) The received signal $R(t)$ in a communication system is given by

$$R(t) = \begin{cases} w(t) + N(t) & \text{if 1 is sent} \\ N(t) & \text{if 0 is sent} \end{cases}$$

where $N(t)$ is white Gaussian noise of spectral density $N_0/2$ and $w(t)$ is as shown below.



At the receiver, the signal $R(t)$ is passed through a filter with impulse response $h(t)$ and the output of the filter is sampled at time t_0 to yield a decision statistic Y. A maximum likelihood decision rule is then used based on Y to decide if 1 or 0 is sent.

(a) (3 pts) For $h(t) = w(4 - t)$, find the error probability if $t_0 = 3$.

(b) (2 pts) Can the error probability in (a) be improved by choosing t_0 differently? Explain your answer.

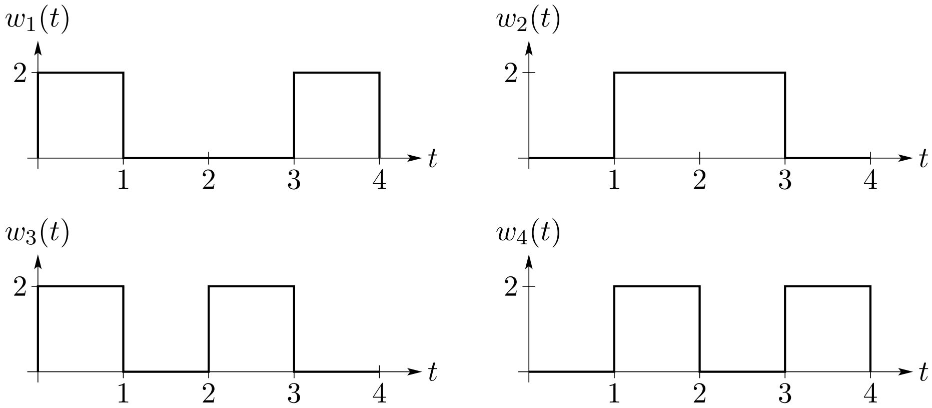
(c) (3 pts) Find the error probability if the filter

$$h(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 2 \\ 0 & \text{else} \end{cases}$$

is employed with $t_0 = 2$.

(d) (3 pts) Can you improve the performance in (c) by using a linear combination of two samples of the filter output rather than using a single sample? Explain.

Problem 3. (12 points) Consider the signal set with four signals as shown below. Each signal is equally likely to be chosen for transmission over an additive white Gaussian noise channel with spectral density $N_0/2$.



(a) (2 pts) Represent the signal set using the four basis signals given by $\psi_1(t) = \text{rect}(t)$, $\psi_2(t) = \text{rect}(t-1)$, $\psi_3(t) = \text{rect}(t-2)$, $\psi_4(t) = \text{rect}(t-3)$ where

$$\text{rect}(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

(b) (2 pts) Use the union bound to find an upper bound to the error probability for the optimal receiver.

(c) (2 pts) Transform the four signals by a translation in order to obtain a minimum energy signal set. Sketch the new signal set $\{\tilde{w}_1(t), \ldots, \tilde{w}_4(t)\}$.

(d) (2 pts) Use the Gram–Schmidt procedure to find an orthogonal basis for the signal set in (c).

(e) (2 pts) Find the exact error probability of the signal set in (c).

(f) (2 pts) Based on your answer to (e) what can you say about the error probability of the receiver in (b)?