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Handout 22 Principles of Digital Communications
Solutions to Problem Set 9 May 5, 2015

Solution 1.

(a)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t)\ dt = \langle \xi(t+\tau), \xi(t)\rangle \\ &\overset{(1)}{\leq} \|\xi(t+\tau)\| \cdot \|\xi(t)\| = \|\xi\| \cdot \|\xi\| = \|\xi\|^2 \overset{(2)}{=} R_\xi(0), \end{aligned}$$

where (1) follows from the Cauchy-Schwarz inequality and (2) from the fact that $R_\xi(0) = \int_{-\infty}^{\infty} \xi(t)\xi^*(t)\ dt = ||\xi||^2$.

(b)

$$\begin{aligned} R_\xi(-\tau) &= \int_{-\infty}^{\infty} \xi(t-\tau)\xi^*(t)\ dt \\ &= \left(\int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau)dt\right)^* \\ &\overset{t\to t+\tau}{=} R_\xi^*(\tau). \end{aligned}$$

(c)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t)\ dt \\ &\overset{t\to t-\tau}{=} \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau)\ dt \\ &= \xi(\tau) \star \xi^*(-\tau). \end{aligned}$$

(d) By Parseval's identity, we have

$$\begin{aligned} R_\xi(\tau) &= \langle \xi(t+\tau), \xi(t)\rangle \\ &= \langle \xi_{\mathcal{F}}(f)e^{\mathrm{j}2\pi f\tau}, \xi_{\mathcal{F}}(f)\rangle \\ &= \int_{-\infty}^{\infty} \xi_{\mathcal{F}}(f)\xi_{\mathcal{F}}^*(f)e^{\mathrm{j}2\pi f\tau}\ df \\ &= \int_{-\infty}^{\infty} |\xi_{\mathcal{F}}(f)|^2 e^{\mathrm{j}2\pi f\tau}\ df, \end{aligned}$$

which is the inverse Fourier transform of $|\xi_{\mathcal{F}}(f)|^2$.

Solution 2.

(a) We have

$$y(t) = \int_{-\infty}^{\infty} w(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of T are

$$\begin{aligned}
y(mT) &= \int_{-\infty}^{\infty} w(\tau)\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \int_{-\infty}^{\infty} \psi(\tau - kT)\psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\} \\
&= d_m.
\end{aligned}$$

(b) Let $\tilde{w}(t)$ be the channel output. Then, $\tilde{y}(t)$ is $\tilde{w}(t)$ filtered by $\psi(-t)$. We have

$$\tilde{w}(t) = w(t) + \rho w(t - T)$$

and

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of T are

$$\begin{aligned}
\tilde{y}(mT) &= \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} [w(\tau) + \rho w(\tau - T)]\psi(\tau - mT)d\tau \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - kT)\right] \psi(\tau - mT)d\tau + \\
&\quad \rho \int_{-\infty}^{\infty} \left[\sum_{k=1}^{K} d_k \ \psi(\tau - T - kT)\right] \psi(\tau - mT)d\tau \\
&= \sum_{k=1}^{K} d_k \mathbb{1}\{k = m\} + \rho \sum_{k=1}^{K} d_k \mathbb{1}\{k = m - 1\} \\
&= d_m + \rho d_{m-1}.
\end{aligned}$$

(c) From the symmetry of the problem, we have

$$P_e = P_e(1) = P_e(-1).$$

$$\begin{aligned}
P_e(1) &= \Pr\left\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = -1\right\} \Pr\left\{D_{k-1} = -1\right\} + \\
&\quad \Pr\left\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = 1\right\} \Pr\left\{D_{k-1} = 1\right\} \\
&= \frac{1}{2}\left(\Pr\left\{Y_k < 0 | D_k = 1, D_{k-1} = -1\right\} + \Pr\left\{Y_k < 0 | D_k = 1, D_{k-1} = 1\right\}\right) \\
&= \frac{1}{2}\left(\Pr\left\{1 - \alpha + Z_k < 0\right\} + \Pr\left\{1 + \alpha + Z_k < 0\right\}\right) \\
&= \frac{1}{2}\left(\Pr\left\{Z_k < -1 + \alpha\right\} + \Pr\left\{Z_k < -1 - \alpha\right\}\right) \\
&= \frac{1}{2}\left[Q\left(\frac{1-\alpha}{\sigma}\right) + Q\left(\frac{1+\alpha}{\sigma}\right)\right].
\end{aligned}$$

SOLUTION 3.

(a) We can easily see that

$$\mathbb{E}\left[X_i | X_{i-1}\right] = \frac{1}{2} X_{i-1} + \frac{1}{2}(-X_{i-1}) = 0.$$

Consequently (using the law of total expectation)

$$\mathbb{E}\left[X_i\right] = \mathbb{E}\left[\mathbb{E}\left[X_i | X_{i-1}\right]\right] = 0.$$

Therefore,

$$K_X[k] = \mathbb{E}\left[(X_i - \mathbb{E}\left[X_i\right])(X_{i-k} - \mathbb{E}\left[X_{i-k}\right])^*\right] = \mathbb{E}\left[X_i X_{i-k}^*\right]$$

Moreover, using the fact that $X_i = X_{i-1} \times (-1)^{D_i}$ repeatedly, we can write

$$X_i = X_{i-k} \times \prod_{j=i-k+1}^{i} (-1)^{D_j}$$

Thus,

$$\begin{aligned}
K_X[k] &= \mathbb{E}\left[X_i X_{i-k}^*\right] \\
&= \mathbb{E}\left[X_{i-k} \prod_{j=i-k+1}^{i} (-1)^{D_j} X_{i-k}^*\right] \\
&\overset{(\star)}{=} \mathbb{E}\left[X_{i-k} X_{i-k}^*\right] \prod_{j=i-k+1}^{i} \mathbb{E}\left[(-1)^{D_j}\right] \\
&= \mathcal{E} \prod_{j=i-k+1}^{i} \mathbb{E}\left[(-1)^{D_j}\right] \\
&= \begin{cases} \mathcal{E} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

where $(\star)$ follows since the data D_i are independent. We have also used the fact that $\mathbb{E}\left[(-1)^{D_i}\right] = 0$.

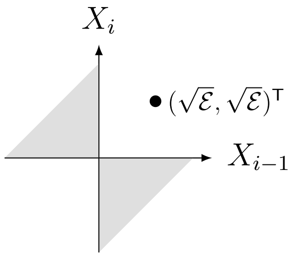
(b) By sampling the signal at the output of the matched filter, $Y(t)$, at multiples of T, we obtain

$$Y(iT) \quad = \quad X_i + Z_i,$$

where Z_i is normally distributed with zero mean and variance $N_0/2$. By looking at the definition of X_i, we see that it is equal to X_{i-1} if $D_i = 0$ and equal to $-X_{i-1}$ if $D_i = 1$. Therefore a simple decoder estimates that $\hat{D}_i = 0$ if Y_i and Y_{i-1} have the same sign, and $\hat{D}_i = 1$ otherwise. This is equivalent to

$$Y_i Y_{i-1} \underset{\hat{D}_i=1}{\overset{\hat{D}_i=0}{\gtrless}} 0.$$

(c) We first compute the error probability when $D_i = 0$. If $X_{i-1} = \sqrt{\mathcal{E}}$, then $X_i = \sqrt{\mathcal{E}}$. When we decode, we will make an error if the signal $(Y_{i-1}, Y_i)^\mathsf{T}$ is in the second or fourth quadrants (shaded regions in the following figure).



Due to the symmetry of the problem, the probability for this to happen is two times the probability for $(Y_{i-1}, Y_i)^\mathsf{T}$ to be in the second quadrant:

$$\Pr\left\{Z_{i-1} < -\sqrt{\mathcal{E}} \cap Z_i > -\sqrt{\mathcal{E}}\right\} = Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right),$$

so,

$$P_e(D_i = 0 | D_{i-1} = 0) = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Again, due to the symmetry of the problem,

$$P_e(D_i = 0 | D_{i-1} = 1) = P_e(D_i = 0 | D_{i-1} = 0) = P_e(D_i = 0),$$

and

$$P_e(D_i = 1) = P_e(D_i = 0);$$

hence

$$P_e = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Solution 4.

(a) In your book, it has been shown that the power spectral density is

$$S_X(f) = \frac{|\psi_{\mathcal{F}}(f)|^2}{T} \sum_k K_X[k] \exp(-\mathrm{j}2\pi k f T),$$

where $K_X[k]$ is the autocovariance of X_i and $\psi_{\mathcal{F}}(f)$ is the Fourier transform of $\psi(t)$.

In this case, because $\{X_i\}_{i=-\infty}^{\infty}$ are i.i.d. and have zero-mean,

$$K_X[k] = \mathbb{E}\left[X_{i+k} X_i^*\right] = \mathcal{E}\mathbb{1}\{k=0\},$$

so

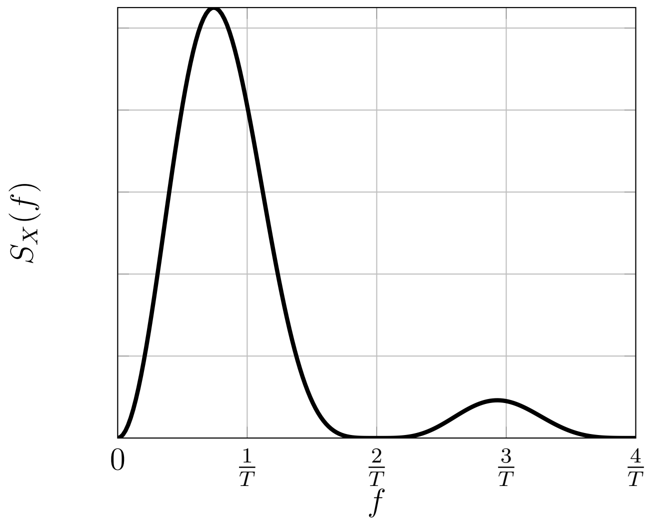
$$S_X(f) = \mathcal{E}\frac{|\psi_{\mathcal{F}}(f)|^2}{T}.$$

$$\begin{aligned}
\psi_{\mathcal{F}}(f) &= \int_{-\infty}^{\infty} \psi(t) e^{-\mathrm{j}2\pi f t}\, dt \\
&= \frac{1}{\sqrt{T}} \int_0^{\frac{T}{2}} e^{-\mathrm{j}2\pi f t}\, dt - \frac{1}{\sqrt{T}} \int_{\frac{T}{2}}^{T} e^{-\mathrm{j}2\pi f t}\, dt \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} \left(e^{-\mathrm{j}2\pi f\frac{T}{2}} - 1 - e^{-\mathrm{j}2\pi f T} + e^{-\mathrm{j}2\pi f\frac{T}{2}}\right) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f\frac{T}{2}} \left(2 - e^{\mathrm{j}2\pi f\frac{T}{2}} - e^{-\mathrm{j}2\pi f\frac{T}{2}}\right) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f\frac{T}{2}} (2 - 2\cos(\pi f T)) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f\frac{T}{2}} 4\sin^2\left(\pi f\frac{T}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_X(f) &= \mathcal{E}\frac{16\sin^4(\pi f\frac{T}{2})}{4\pi^2 f^2 T^2} \\
&= \mathcal{E}\left(\pi f\frac{T}{2}\right)^2 \operatorname{sinc}^4\left(f\frac{T}{2}\right).
\end{aligned}$$

We first see that $S_X(0) = 0$. Furthermore, for $f > 0$, $S_X(f) = 0$ if and only if $\sin(\pi f\frac{T}{2}) = 0$. That is $f\frac{T}{2} = m \in \mathbb{Z}$ which means the spectrum is zero at frequencies $f_m = m\frac{2}{T}$, $m \in \mathbb{Z}$ (multiples of $\frac{2}{T}$). $S_X(f)$ is plotted here:



(b) Using the described precoding,

$$\begin{aligned}
K_X[k] &= \mathbb{E}\left[X_{i+k}X_i^*\right] \\
&= \mathcal{E}\,\mathbb{E}\left[(D_{i+k}+\alpha D_{i+k-1})(D_i+\alpha D_{i-1})\right] \\
&= \mathcal{E}\left(\mathbb{E}\left[D_{i+k}D_i\right]+\alpha\mathbb{E}\left[D_{i+k}D_{i-1}\right]+\alpha\mathbb{E}\left[D_{i+k-1}D_i\right]+\alpha^2\mathbb{E}\left[D_{i+k-1}D_{i-1}\right]\right) \\
&= \mathcal{E}\left(\mathbb{1}\{k=0\}+\alpha\mathbb{1}\{k=-1\}+\alpha\mathbb{1}\{k=1\}+\alpha^2\mathbb{1}\{k=0\}\right) \\
&= \begin{cases} (1+\alpha^2)\mathcal{E} & k=0, \\ \alpha\mathcal{E} & k=\pm 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus,

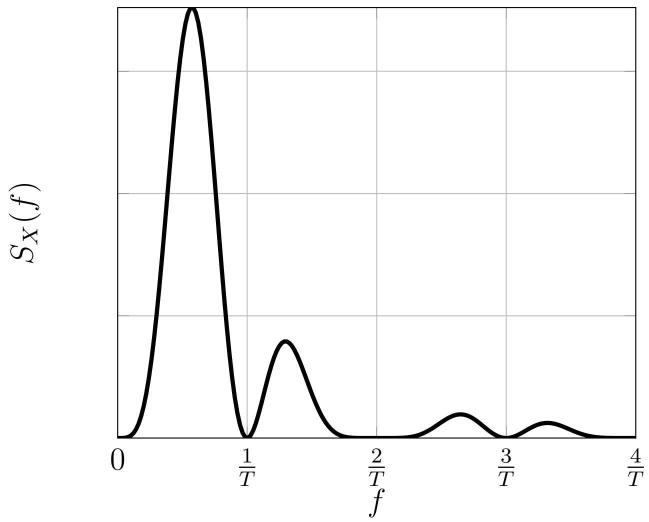
$$\begin{aligned}
\sum_k K_X[k]\exp(-\mathtt{j}2\pi kfT) &= \left((1+\alpha^2)+\alpha\exp(\mathtt{j}2\pi fT)+\alpha\exp(-\mathtt{j}2\pi fT)\right)\mathcal{E} \\
&= (1+\alpha^2+2\alpha\cos(2\pi fT))\mathcal{E} \qquad (1)
\end{aligned}$$

To create a null at $f=\frac{1}{T}$, the above should evaluate to 0 at $f=\frac{1}{T}$. That is,

$$1+\alpha^2+2\alpha=(1+\alpha)^2=0.$$

Thus, by choosing $\alpha=-1$ we can create the desired zero at $f=\frac{1}{T}$. In this case the power spectrum of $X(t)$ will be

$$\begin{aligned}
S_X(f) &= \mathcal{E}\frac{16\sin^4(\pi f\frac{T}{2})}{4\pi^2f^2T^2}\left(2-2\cos(2\pi fT)\right) \\
&= \mathcal{E}\frac{64\sin^4(\pi f\frac{T}{2})}{4\pi^2f^2T^2}\sin^2(\pi fT)
\end{aligned}$$



(c) The precoding of part (b) cannot create the desired nulls since for (1) to evaluate to 0 at $f = m\frac{1}{4T}$, we must have

$$1 + \alpha^2 + 2\alpha\cos(m\pi/2) = 0 \qquad \forall m \in \mathbb{Z}.$$

However, this is impossible since for m odd we require $1 + \alpha^2 = 0$ which cannot be satisfied for any real α.

However, one can create nulls at multiples of $\frac{1}{4T}$ using a precoding of the form

$$X_i = \sqrt{\mathcal{E}}(D_i + \alpha D_{i-4}).$$

Using such a precoding, following the same steps as in (b) we can conclude that

$$K_X[k] = \begin{cases} (1+\alpha^2)\mathcal{E} & k = 0, \\ \alpha\mathcal{E} & k = \pm 4, \\ 0 & \text{otherwise.} \end{cases}$$

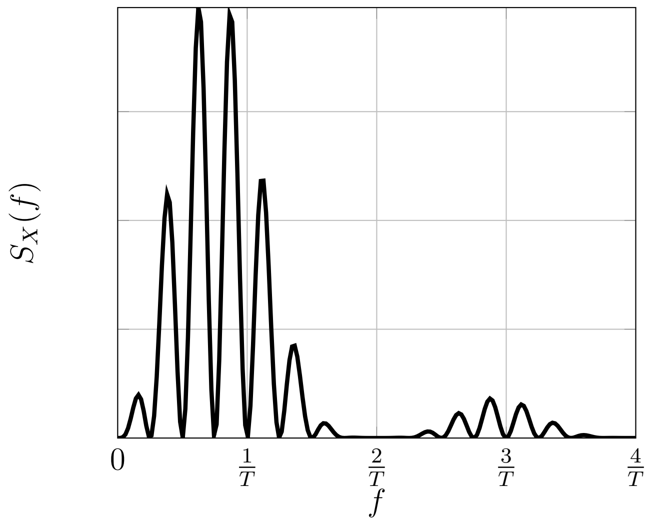
Therefore,

$$\begin{aligned} \sum_k K_X[k]\exp(-\mathtt{j}2\pi kfT) &= \big((1+\alpha^2) + \alpha\exp(\mathtt{j}2\pi 4fT) + \alpha\exp(-\mathtt{j}2\pi 4fT)\big)\,\mathcal{E} \\ &= (1 + \alpha^2 + 2\alpha\cos(8\pi fT))\mathcal{E} \end{aligned} \tag{2}$$

which evaluates to 0 at $f = m\frac{1}{4T}$ for $\forall m \in \mathbb{Z}$ if we choose $\alpha = -1$.

In this case, the power spectrum of $X(t)$ will similarly be

$$S_X(f) = \mathcal{E}\frac{64\sin^4(\pi f\frac{T}{2})}{4\pi^2 f^2 T^2}\sin^2(4\pi fT)$$



Solution 5. From Theorem 5.6, we know that $\{\psi(t-jT)\}_{j=-\infty}^{\infty}$ is an orthonormal set if and only if

$$\sum_{k\in\mathbb{Z}} |\psi_{\mathcal{F}}(f-\frac{k}{T})|^2 = T.$$

(a)

$$\sum_{k\in\mathbb{Z}} T\mathbb{1}_{\left[\frac{k}{T}-\frac{1}{2T},\frac{k}{T}+\frac{1}{2T}\right]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

$\Rightarrow \quad \psi(t)$ is orthonormal to its time-translates by multiples of T.

(b)

$$\sum_{k\in\mathbb{Z}} \frac{T}{2}\mathbb{1}_{\left[\frac{k-1}{T},\frac{k+1}{T}\right]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

$\Rightarrow \quad \psi(t)$ is orthonormal to its time-translates by multiples of T.

(c) Because $|\psi_{\mathcal{F}}(f)|^2$ vanishes outside $\left[-\frac{1}{T},\frac{1}{T}\right]$, we verify whether the band-edge symmetry is fulfilled, which is the case. Hence, the Nyquist criterion is satisfied and $\psi(t)$ is orthonormal to its time-translates by multiples of T. Note: the same reasoning can be applied to (b).

(d) $\psi_{\mathcal{F}}(f)$ is a sinc function, therefore $\psi(t)$ is a box function, equal to $\frac{1}{T}\mathbb{1}_{\left[-\frac{T}{2},\frac{T}{2}\right]}(t)$. This is orthogonal to its time-translates by multiples of T, but does not have unit norm (unless $T=1$): $\int_{-\infty}^{\infty} |\psi(t)|^2 \, dt = \frac{1}{T}$.